Totally Geodesic Foliations and sub-Riemannian Geometry

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ABSTRACT

In this thesis, we examine the theory of sub-Riemannian geometries arising as transversal distributions to totally geodesic foliations. Various connections from the literature are examined, and their adaptedness to the foliation structure and suitability for computation of variational problems is discussed. We study particularly the notion of H-type foliation that was jointly introduced in [24]. A generalized curvature dimension inequality, horizontal Einstein property, and classification result are achieved. The holonomy of H-type foliations is explored, in particular we achieve a result on the holonomy of H-type submersions. Finally comparison theorems for the Hessian and Laplacian of the distance function based on variational principles are presented from the joint work [25].

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Chapter 1

Introduction

This thesis explores sub-Riemannian geometries that can arise as transversal distributions to Riemannian foliations. This notion was originally motivated as a broad generalization of Kaplan's H-type groups, where one considers the essential structure to be the horizontal action of Clifford module generated by the vertical space. On an appropriately compatible Riemannian foliation we can define the notion of an H-type foliation, where the Clifford algebra generated by the vertical distribution gives significant information about the sub-Riemannian geometry; these objects encompass a number of important examples that are thereby justified as models for comparison theorems.

The thesis is structured as follows:

Chapter 1: We review relevant notions of foliation theory and sub-Riemannian geometry, as well as introduce the main results of the thesis.

Chapter 2: We study the theory of connections adapted to foliations. In particular we consider the conditions under which certain well-established connections in the literature coincide, and the theory of applications of these connections to geodesics and Jacobi fields.

Chapter 3: We present H-type foliations, a generalization of Kaplan's H-type groups, as introduced in [24]. By considering a complementary Riemannian foliation on these spaces we deduce structural results on the sub-Riemannian geometry of interest.

Chapter 4: We study a notion of horizontal holonomy on H-type foliations that is well defined by considering an adapted connection to the foliation. In particular we recover a relationship between horizontal holonomy of H-type submersions and the classification of nonsymmetric Riemannian holonomies due to Berger-Simons-Olmos.

Chapter 5: We present comparison theorems on H-type foliations determined by considering the convergence of Riemannian penalty metrics, as in [25]. A Bonnet-Meyers type diameter bound and sub-Laplacian comparison that classically follow from Ricci curvature lower bounds are recovered.

1.1 Background

Throughout the thesis we will assume a familiarity with differential geometry, especially the theory of Riemannian manifolds. In this section we will review core concepts of the theory of foliations and of sub-Riemannian geometry because of their central nature to the topic of the thesis and to set notation.

1.1.1 Foliations

A foliation is a partition of a manifold into equivalence classes that locally models the partition of R^{n+m} by submanifolds R^m.

Definition 1.1.1. Let $\mathbb{M}$ be a $n+m$ dimensional manifold. Suppose there exists a disjoint collection $\mathcal{F}$ of connected, immersed m-dimensional submanifolds such that for each $p \in \mathbb{M}$ there is a neighborhood U_p and a smooth submersion

$$\phi_{U_p} \colon U_p \to \mathbb{R}^n$$

with the property that for any $x \in \mathbb{R}^n$ the set $f^{-1}(x)$ is either empty or the intersection of one of the submanifolds of $\mathcal{F}$ with U_p.

We call the collection $\mathcal{F}$ a (codimension-n) foliation of $\mathbb{M}$, and the submanifolds leaves. A primary resource and overview of the extensive literature on foliations is [111]. Other valuable references include [66, 73, 87, 30, 86]

Remark 1.1.2. It is important to note that foliations are locally modeled by submersions; this will be essential multiple times in the sequel.

In particular, we are interested in the structure of the tangent spaces to a foliation. On a manifold $\mathbb{M}$ with foliation $\mathcal{F}$ there is a natural subbundle $\mathcal{V}$ of $T\mathbb{M}$ defined by the property that at every point $p \in \mathbb{M}, \mathcal{V}_p$ is the tangent space of the leaf of $\mathcal{F}$; we will refer to $\mathcal{V}$ as the vertical distribution associated to the foliation. By the Frobenius integrability theorem [58] $\mathcal{V}$ is completely integrable, by which we mean that for any vector fields $X, Y \in \mathcal{V}$ it must hold that the Lie bracket $[X, Y] \in \mathcal{V}$.

Given a Riemannian metric g on a foliation $(\mathbb{M}, \mathcal{F})$ we can consider the transverse distribution $\mathcal{H}$ defined such that at every point $p \in \mathbb{M}$ the tangent space splits orthogonally as

$$T_p\mathbb{M} = \mathcal{H}_p \oplus \mathcal{V}_p.$$

Conversely, given a distribution $\mathcal{H}$ with the property that the intersection $\mathcal{H}_p \cap \mathcal{V}_p$ is trivial for all $p \in \mathbb{M}$ we have up to a choice of normalization a Riemannian metric g that will orthogonally split the tangent bundle. It is important for our purpose to understand that this construction does not impose any condition on the integrability of $\mathcal{H}$. We will call a foliation equipped with a Riemannian metric a Riemannian foliation.

We will want to distinguish certain geometric conditions on Riemannian foliations.

(a) We will say that $(\mathbb{M}, \mathcal{F}, g)$ is totally-geodesic if all of the leaves of $\mathcal{F}$ are totally-geodesic submanifolds; that is if every geodesic of the leaves is a geodesic of $\mathbb{M}$. See especially [73].

(b) We will say that $(\mathbb{M}, \mathcal{F}, g)$ is has bundle-like metric if the local submersions of the foliation are diffeomorphisms.

Remark 1.1.3. We mention that these conditions do impose conditions on the curvature of $\mathbb{M}$, the dimensions of $\mathcal{H}, \mathcal{V}$, and on the integrability of $\mathcal{H}$, as shown in [31]. In particular, $\mathcal{H}$ cannot be completely integrable.

It is a result of Tondeur that these properties can be characterized by the Lie derivative of the metric.

Theorem 1.1.4 (Tondeur [110])**.** *Let* $(\mathbb{M}, \mathcal{F}, g)$ *be a Riemannian foliation.*

(a) $(\mathbb{M}, \mathcal{F}, g)$ *is totally-geodesic if and only if for all* $X \in \mathcal{H}, Z \in \mathcal{V}$ *it holds that*

$$(\mathcal{L}_X\, g)(Z, Z) = 0.$$

(b) $(\mathbb{M}, \mathcal{F}, g)$ *has bundle-like metric if and only if for all* $X \in \mathcal{H}, Z \in \mathcal{V}$ *it holds that*

$$(\mathcal{L}_Z \, g)(X, X) = 0.$$

Unless otherwise stated, we will always insist that our foliations be totally geodesic and have bundle-like metric.

1.1.2 Sub-Riemannian Geometry

Sub-Riemannian geometry is the study of manifolds that allow for a notion of motion or length as in Riemannian geometry, but in a constrained way.

Definition 1.1.5. Suppose $\mathbb{M}$ is a smooth manifold. If $\mathcal{H}$ is a subbundle of $T\mathbb{M}$ that has the property that at every point $p \in \mathbb{M}$ the entire tangent space $T_p\mathbb{M}$ is generated by finitely many brackets of vectors in $\mathcal{H}_p$, we say $\mathcal{H}$ is bracket-generating. If moreover $(\mathbb{M}, \mathcal{H})$ is equipped with a fiberwise inner-product $g_{\mathcal{H}}$, we say the triple $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold with horizontal distribution $\mathcal{H}$.

The smallest number of vector fields $x_1, x_2, \ldots, x_r \in \mathcal{H}_p$ needed to generate $T_p\mathbb{M}$ is called the step of the structure at p. If this is constant across $\mathbb{M}$, we say the sub-Riemannian structure is equivariant.

Sub-Riemannian geometry is increasingly of broad research interest, both intrinsically and as it arises in many natural situations such as in physics, control theory, PDEs, stochastic differential equations, and many other fields. Some excellent references include [88, 7, 2, 32, 46, 70, 101]. We will work always with equivariant sub-Riemannian manifolds of step 2 in this thesis; it is an interesting direction of research to consider generalizations of our results to higher step.

It is immediately clear that $\mathcal{H}$ is not integrable; in fact the bracket-generating condition is equivalent to $\mathcal{H}$ being as far from integrable as possible. This allows for sub-Riemannian manifolds to be complete, in the following sense.

Definition 1.1.6. Suppose $\gamma\colon [0,1] \to \mathbb{M}$ is a smooth curve with the property $\dot{\gamma}(t) \in \mathcal{H}_{\gamma}(t)$ for almost every $0 \leq t \leq 1$. We say that γ is horizontal, and we define its length

$$\ell(\gamma) = \int_0^1 \sqrt{g_{\mathcal{H}}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt$$

Moreover, for any two points $p, q \in \mathbb{M}$ we define the Carnot-Caratheodory distance d_{cc} by

$$d_{cc}(p,q) = \inf_{\gamma \in C(p,q)} \ell(\gamma)$$

where $C(p,q)$ is the collection of horizontal curves connecting p to q.

Importantly, we have a result on the completeness of the metric.

Theorem 1.1.7 (Chow [51], Rashevskii [98]). *On a sub-Riemannian manifold* $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ *the bracket-generating condition implies that any two points* p, q *can be connected by an almost everywhere horizontal curve of finite length.*

Many sub-Riemannian manifolds have a natural foliation structure. That is, there exist Riemannian foliations $(\mathbb{M}, \mathcal{F}, g)$ such that the metric splits orthogonally as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$ where $\mathcal{V}$ is the (completely integrable) tangent distribution to the leaves and the transversal distribution $\mathcal{H}$ is bracket-generating, thus the triple $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold. These will be primary objects of interest and we will see many examples, particularly in section 3.3. We refer to [26, 22, 63, 64] for more about the sub-Riemannian geometry associated to foliations.

1.2 Motivating Questions

Sub-Riemannian geometry is a relatively young field and many important questions remain open. Much of the approach to the field begins in analogy to Riemannian geometry: recalling important Riemannian results, do they carry forward (with some suitable generalization) to sub-Riemannian structures?

1.2.1 Curvature in sub-Riemannian geometry

Definition 1.2.1. Let $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ be a sub-Riemannian manifold, and let $(\mathbb{M}, g)$ be a Riemannian manifold such that $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$ is an orthogonally splitting extension of $g_{\mathcal{H}}$. We define the associated penalty metric

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$$

Clearly, $g_1 = g$, and for any $\varepsilon > 0$ the pair $(\mathbb{M}, g_\varepsilon)$ is a Riemannian manifold. As $\varepsilon \to 0^+$ the magnitude of any vertical vector approaches $+\infty$; heuristically we can interpret this as the "cost" to move in a vertical direction as increasing without bound and so in the limit the only curves which will have finite length are those that are everywhere tangent to the horizontal distribution. This is made precise in the following sense:

Theorem 1.2.2. *In the Gromov-Hausdorff sense we have the convergence*

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$$

From here, one could hope to directly recover many classical Riemannian results on sub-Riemannian manifolds by consideration of the limit of Riemannian curvature. Unfortunately this isn't possible per the following lemma.

Lemma 1.2.3. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with penalty metric. Denote by* $\mathrm{Ric}^\varepsilon(X,Y) = \mathrm{Tr}\, g_\varepsilon(R^\varepsilon(X,\times)\times, Y)$ *the Ricci curvature associated to the metric* g_ε *with Levi-Civita connection* ∇^ε*,*

$$\lim_{\varepsilon\to 0^+} \mathrm{Ric}^\varepsilon(X,Y) = \begin{cases} -\infty & X,Y \in \mathcal{H} \\ +\infty & X,Y \in \mathcal{V} \end{cases}$$

As a consequence, any Riemannian result that relies on lower curvature bounds will fail. Much recent literature has been dedicated to resolving this, and in particular there is significant work in determining an appropriate definition of curvature quantities in the sub-Riemannian setting in order to recover Riemannian-type results. Arguably there are two main schools of thought:

- Hamiltonian, as developed in [85], in which one considers the intrinsic sub-Riemannian Hamiltonian on the cotangent bundle, and thereby studies variational problems.

- Eulerian, as developed in [21], in which one allows for an analytic structure defined on a complementary distribution $\mathcal{V}$ and arrives at purely sub-Riemannian results by showing an independence from the choice of complement.

One primary motivation for our study of H-type foliations is an agreement of these methods; as we will see, there is a sense in which results from both schools are meaningful in this setting and for which the results are complementary.

1.2.2 Models Spaces in sub-Riemannian Geometry

In particular, there is a notion of *comparison theory* in Riemannian geometry. It is well established that the only Riemannian manifolds of constant curvature are the sphere S^n, the Euclidean space $\mathbb{R}^n$, and the hyperbolic space $\mathbf{H}^n$ with positive, zero, and negative curvature, respectively. On these spaces one computes quantities of interest explicitly, and then establishes results that determine conditions under which these quantities can be compared to those of the model spaces. This process includes results such as

- Rauch and Laplacian comparison theorem
- Bonnet-Meyers diameter and compactness theorem
- Bishop-Gromov inequality
- Cheng rigidity theorem
- Eigenvalue estimates

among others. Leaving aside the issue of determining a precise notion of curvature, there is a growing consensus [16] that among step 2 sub-Riemannian structures that the Hopf fibration, the Heisenberg group, and the Anti-de Sitter fibration are appropriate models analogous to the the Riemannian ones for comparison theorems to be built upon. Key properties of these models are captured by the notion of H-type foliation that we examine.

Remark 1.2.4. The issue becomes significantly more difficult in higher step, as it becomes apparent that any single curvature quantity is insufficient to determine model spaces, see for example [61] and the references therein.

1.3 Main Results

In Chapter 2, our main goal is to understand the notion of a connection adapted to a foliation (definition 2.2.1) and to explore examples of this in the literature. In particular, we describe the Bott connection (theorem 2.2.2) axiomatically and provide a series of results describing the equivalence of other connections with the Bott connection.

Structure	Torsion	Reference
Complex Type, $m = 1, n = 2k$		
K-Contact	YM	[3] [28]
Sasakian	CP	[3] [40]
Heisenberg Group	CP	[45]
Hopf Fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2k+1} \to \mathbb{C}P^k$	CP	[27]
Anti de-Sitter Fibration $\mathbb{S}^1 \hookrightarrow \mathbf{AdS}^{2k+1}(\mathbb{C}) \to \mathbb{C}H^k$	CP	[43] [112]
Twistor Type, $m = 2, n = 4k$		
Twistor space over quaternionic Kähler manifold	HP	[65] [105]
Projective Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}P^{2k+1} \to \mathbb{H}P^k$	HP	[29]
Hyperbolic Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}H^{2k+1} \to \mathbb{H}H^k$	HP	[20] [43]
Quaternionic Type, $m = 3, n = 4k$		
3K-contact	YM	[72] [109]
Negative 3K-contact	YM	[72] [109]
3-Sasakian	HP	[39] [104]
Negative 3-Sasakian	HP	[39]
Torus bundle over hyperkähler manifolds	CP	[67]
Quaternionic Heisenberg Group	CP	[45]
Quaternionic Hopf Fibration $\mathbf{SU}(2) \hookrightarrow \mathbb{S}^{4k+3} \to \mathbb{H}P^k$	HP	[29]
Quaternionic Anti de-Sitter Fibration $\mathbf{SU}(2) \hookrightarrow \mathbf{AdS}^{4k+3}(\mathbb{H}) \to \mathbb{H}H^k$	HP	[20] [43]
Octonionic Type, $m = 7, n = 8$		
Octonionic Heisenberg Group	CP	[45]
Octonionic Hopf Fibration $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{O}P^1$	HP	[94]
Octonionic Anti de-Sitter Fibration $\mathbb{S}^7 \hookrightarrow \mathbf{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1$	HP	[43]
H-type Groups, m is arbitrary	CP	[52] [74]

Table 1.3.1: [24, Table 3] Some examples of H-type foliations.

In Chapter 3 we follow [24]. The primary object of the thesis, H-type foliations (definition 3.2.2), are justified and introduced. To indicate the breadth of these objects, we reproduce table 1.3.1 [24, Table 3], classified by the behavior of the Bott torsion (definition 3.2.4).

We show that H-type foliations are Yang-Mills (lemma 3.2.24) and thereby satisfy a generalized curvature dimension inequality (theorem 3.2.23) under a Ricci curvature condition on the horizontal distribution. By consideration of parallel Clifford structures 3.2.2, there is a achieved a complete classification of H-type foliations arising as global submersions section 3.2.1. We also establish an horizontal Einstein property (theorem 3.4.7) giving the necessary curvature bounds so that on a wide class of H-type foliations we have the following result.

Theorem 1.3.1 (theorem 3.4.12, [24, Corollary 3.20])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with a parallel horizontal Clifford structure such that* $\kappa > 0$*. Then,* $\mathbb{M}$ *is compact with finite fundamental group. Moreover,*

- *If* $m \neq 3$ *or* $m = 3$ *and* $(\mathbb{M}, \mathcal{H}, g)$ *is of quaternionic type then we have the sub-Riemannian diameter bound*

$$\mathrm{diam}(\mathbb{M}, d_{cc}) \leq 4\sqrt{3}\frac{\pi}{\sqrt{\kappa}}\sqrt{\frac{(n+4m)(n+6m)}{n(n+8(m-1))}},$$

  and we have the following estimate for the first eigenvalue of the sub-Laplacian

$$\lambda_1 \geq \frac{\kappa}{4}\frac{n(n+8(m-1))}{n+3m-1}.$$

- *If* $m = 3$ *and* $(\mathbb{M}, \mathcal{H}, g)$ *is not of quaternionic type, then we have the sub-Riemannian diameter bound*

$$\mathrm{diam}(\mathbb{M}, d_{cc}) \leq 2\sqrt{6}\frac{\pi}{\sqrt{\kappa}}\sqrt{\frac{(n+12)(n+18)}{n(n+8)}},$$

and we have the following estimate for the first eigenvalue of the sub-Laplacian

$$\lambda_1 \geq \frac{n\kappa}{2}.$$

In Chapter 4 we explore a notion of horizontal holonomy of H-type foliations. In particular, we show that on H-type submersions there is a strong relationship between the horizontal holonomy and the Riemannian holonomy of the base space.

Theorem 1.3.2 (theorem 4.3.7)**.** *For an H-type submersion* $(\mathbb{M}, \mathcal{H}, g, \pi)$*,*

$$\mathbf{Hol}^0(\mathcal{H}) \cong \overline{\mathbf{Hol}}^0(\mathbb{B})$$

We also achieve a structural theorem in the more general setting of H-type foliations with parallel horizontal Clifford structure (definition 3.2.21).

Theorem 1.3.3 (theorem 4.3.11)**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with parallel horizontal Clifford structure, and set* $n = \operatorname{rank}(\mathcal{H}), m = \operatorname{rank}(\mathcal{V})$*.*

(a) *If* $m = 1$*, then* $\mathbf{Hol}^0(\mathcal{H})$ *is isomorphic to a subgroup of* $\mathbf{U}(n/2)$*.*

(b) *If* $m \geq 2$ *and* $\kappa = 0$*, then* $\mathbf{Hol}^0(\mathcal{H})$ *is isomorphic to a subgroup of* $\mathbf{Sp}(n/4)$*.*

(c) *If* $m = 3$ *and and the maps* $J_z, z \in \mathcal{V}$ *form a Lie algebra under commutation at every point, then* $\mathbf{Hol}^0(\mathcal{H})$ *is isomorphic to a subgroup of* $\mathbf{Sp}(1)\mathbf{Sp}(n/4)$

In Chapter 5 we follow [25], in which we consider a family of Riemannian metrics on an H-type foliation converging to the sub-Riemannian structure. By consideration of Jacobi fields for adapted metric connections with metric adjoint (section 2.3) we are able to establish uniform comparison theorems that thereby hold in the sub-Riemannian limit. These include

Theorem 1.3.4 (theorem 5.3.11, [25, Theorem 3.10(b)]). *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation that is complete and has horizontally parallel torsion. Assume there is some* $\rho > 0$ *such that for any unit* $X \in \mathcal{H}, Z \in \mathcal{V}$ *we have*

$$\mathrm{Sec}(X \wedge J_Z X) \geq \rho.$$

Then

$$\mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\rho}}$$

Theorem 1.3.5 (theorem 5.3.15, [25, Theorem 3.12]). *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with parallel horizontal Clifford structure satisfying the* J^2 *condition. Let* $x \in \mathbb{M}$ *and define* $r_0(y) = d_0(x, y)$. *Assume there exists* $\rho > 0$ *such that*

$$\mathrm{Sec}(X \wedge Y) \geq \rho$$

for all $X, Y \in \mathcal{H}$. *For* $y \notin \mathbf{Cut}_0(x)$ *we have*

$$\Delta_{\mathcal{H}} r_0 \leq (n - m - 1) F_{\mathrm{Riem}}(r_0, K_{\mathrm{Riem}}) + F_{\mathrm{Sas}}(r_0, K_{\mathrm{Sas},\dot{\gamma}}) + (m - 1) F_{\mathrm{Sas}}(r_0, K_{\mathrm{Sas},\perp})$$

where

$$K_{\mathrm{Riem}} = \rho + \frac{1}{4}\|\nabla_{\mathcal{V}} r_0\|^2$$
$$K_{\mathrm{Sas},\dot{\gamma}} = \rho + \|\nabla_{\mathcal{V}} r_0\|^2$$
$$K_{\mathrm{Riem},\perp} = \rho - 2\|\nabla_{\mathcal{V}} r_0\|^2.$$

Chapter 2

Connections on Foliations

In this chapter we examine the notion of connections on foliations; in particular we investigate what it means for a connection to be adapted to a metric and to a foliation, and present a series of useful results for connections with a variety of properties.

2.1 Theory of Connections

Let $\mathbb{M}$ be a smooth manifold. There exists an intrinsic notion of differentiation of vector fields on $\mathbb{M}$ given by the Lie derivative, defined for vector fields X, Y on $\mathbb{M}$ as the derivation

$$\mathcal{L}_X Y = [X, Y] = XY - YX.$$

This follows from the perspective of the Lie derivative as the appropriate first-order term in the flow generated by a vector field, that is

$$\mathcal{L}_X Y|_p = \lim_{t \to 0} \frac{DF^{-t}(Y|_{F^t(p)}) - Y|_p}{t}$$

where $t \mapsto F^t(0)$ is the flow generated by X, that is the integral curve for X with $F^0(p) = p$. We refer to standard texts such as [113, 77, 80] for more details.

Unfortunately, it isn't difficult to see that this definition depends on X not only at p, but in a neighborhood of p; this is undesirable from the point of view of parallel transport, as it's not possible to sensibly describe the transport of a vector field along a curve.

One resolution of this issue with the Lie derivative arises in the notion of a connection.

Definition 2.1.1. Let $\pi\colon E \to \mathbb{M}$ be a vector bundle over $\mathbb{M}$, and suppose

$$\nabla\colon \Gamma(T\mathbb{M}) \otimes \Gamma(E) \to \Gamma(E)$$

written $(X, s) \mapsto \nabla_X s$ is a map such that

1. For a fixed $s, X \mapsto \nabla_X s$ is a $(1,1)$ tensor. That is

   $$\nabla_{fv+gu} s = f\nabla_v s + g\nabla_u s$$

   for vectors $u, v \in T_p\mathbb{M}$ and functions f, g.

2. For a fixed $X, s \mapsto \nabla_X s$ is a derivation. That is

   $$\nabla_X(fs + t) = (Xf)s + f\nabla_X s + \nabla_X t$$

   for sections $s, t \in \Gamma(E)$ and functions f.

Then we call ∇ an <u>(affine) connection</u> on E.

Remark 2.1.2. Unlike for Lie derivatives, it follows from the tensorial property 1 that the map $X \mapsto \nabla_X Y$ at a point $p \in \mathbb{M}$ depends only on $X|_p$, and so it is sensible to understand connections as a form of directional derivative.

There are many definitions of connections in the literature, see for example [35, 77, 96] for thorough introductions.

Remark 2.1.3. A connection defined on the tangent bundle $\pi \colon T\mathbb{M} \to \mathbb{M}$ can be extended to a connection on all tensor fields by requiring that a Liebniz' rule and product rule hold. Specifically, for an (s, r)-tensor $S = S_1 \otimes \cdots \otimes S_s$, we require the Liebniz' rule

$$(\nabla_X S)(Y_1, \ldots, Y_r) = \nabla_X (S(Y_1, \ldots, Y_r)) - \sum_{i=1}^{r} S(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_r)$$

and the product rule

$$\nabla_X S = \sum_{i=1}^{s} S_1 \otimes \cdots \otimes (\nabla_X S_i) \otimes \cdots \otimes S_s.$$

We will also define a $(s, r+1)$-tensor ∇S by

$$(\nabla S)(X, Y_1, \ldots, Y_r) = (\nabla_X S)(Y_1, \ldots, Y_r).$$

We will be concerned primarily with such connections, and will refer to a connection defined on $T\mathbb{M}$ as a <u>connection on $\mathbb{M}$</u>. Hereafter all connections should be assumed to be connections on $\mathbb{M}$ unless otherwise stated.

Connections are not intrinsic to the structure of a manifold, in the sense that any manifold will have many connections. In fact, we can characterize all connections on a manifold as differing by a tensor by the following.

Theorem 2.1.4. *Suppose* ∇^1, ∇^2 *are both connections on* $\mathbb{M}$*. Then*

$$A(X,Y) = \nabla^1_X Y - \nabla^2_X Y$$

is a $(1,2)$*-tensor. Moreover, for any connection* ∇^1 *on* $\mathbb{M}$ *and any* $(1,2)$*-tensor* A*,*

$$\nabla^2_X Y = \nabla^1_X Y + A(X,Y)$$

is a connection on $\mathbb{M}$*.*

Proof. We refer to [35] for the proof. □

Up to the existence of at least one connection, we can see that there is a bijection from the set of $(1,2)$-tensor fields on $\mathbb{M}$ to the set of connections on $\mathbb{M}$.

Remark 2.1.5. Because of this, we want to emphasize the perspective that connections are extrinsic, and that any results on the topological, smooth, Riemannian, or other structures on a manifold should be independent of the choice of connection. They should be considered tools for the computation of intrinsic results. However, the choice of a connection can be a powerful tool to simplify computations.

2.1.1 Parallel Transport

Improving on the situation with the Lie derivative, connections induce a notion of parallel transport of tensors on a manifold. In particular, we say that a tensor s along a curve $\gamma\colon [0,T] \to \mathbb{M}$ is ∇-parallel if

$$\nabla_{\dot\gamma} s = 0.$$

This is well-defined since connections are tensorial in the first component. In fact, given a curve γ we sometimes define the covariant derivative along γ,

$$D_t s = \left(\nabla_{\dot\gamma(t)} s\right)(\gamma(t)).$$

This is only well-defined for s defined on a neighborhood of γ, but we are usually interested in results that are independent of a choice of extension for $s|_\gamma$.

If one has a curve $\gamma\colon [0,1] \to \mathbb{M}$ and a tensor $s \in E_{\gamma(0)}$, one can define a tensor field s along γ by parallel transport. This is an ODE and has has a unique solution, which we will refer to as the parallel transport of s along γ.

We will also make sense of the notion of a ∇-parallel structure in the following way. Suppose S is a subspace of the total space E of a vector bundle $\pi\colon E \to \mathbb{M}$. If S has the property that for all $s \in S$ and any vector field $X\Gamma(T\mathbb{M})$ that

$$\nabla_X s \in S$$

then we say S is ∇-parallel.

2.1.2 Levi-Civita Connection

For any connection ∇ we can define the torsion tensor

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

We will say that ∇ is torsion-free if $T^{\nabla} \equiv 0$.

On a Riemannian manifold $(\mathbb{M}, g)$ we compute that

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

and we will say that ∇ is compatible with g or that it is metric if $\nabla g \equiv 0$; this is motivated by the fact that for metric connections ∇ we have the sensible formula for the derivative of the metric

$$X \cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Equipped with these definitions, there is the well-known

Theorem 2.1.6 (Fundamental Theorem of Riemannian Geometry)**.** *Let* $(\mathbb{M}, g)$ *be a Riemannian manifold. Then there exists a unique connection* ∇^g *on* $\mathbb{M}$ *that is metric and torsion-free. We call* ∇^g *the Levi-Civita connection.*

Proof. Suppose that ∇ is a metric and torsion-free connection on $\mathbb{M}$. We have the metric relation

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y,Z)$$

which we can cyclically alternate to recover the Koszul formula

$$2g(\nabla_X Y, Z) = Yg(X,Z) + Xg(Z,Y) - Zg(Y,X) \\ - g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X).$$

This formula uniquely determines ∇^g, proving both existence and uniqueness. □

2.1.3 Metric connections and the Koszul formula

By theorem 2.1.4 any connection on a Riemannian manifold can be written as the sum of the Levi-Civita connection with a $(1,2)$-tensor, but it will often be more useful to understand connections by an axiomatic description. From the proof of the existence and uniqueness of the Levi-Civita connection, we can make the following observation.

Theorem 2.1.7. *Let ∇ be a metric (but not necessarily torsion-free) connection on a Riemannian manifold. Then we can write*

$$\nabla_X Y = \nabla^g_X Y + A(X,Y)$$

where

$$A^\nabla(X,Y) = \frac{1}{2}\left(T^\nabla(X,Y) - J^\nabla_Y X - J^\nabla_X Y\right)$$

and J^∇ is given by

$$J^\nabla_X Y = ((T^\nabla)^\flat(Y,\cdot,X))^\sharp. \tag{2.1.1}$$

Proof. The metric relation

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y,Z)$$

can be cyclically summed to recover the general Koszul formula

$$\begin{aligned}2g(\nabla_X Y, Z) = Yg(X,Z) + Xg(Z,Y) - Zg(Y,X) \\ - g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X) \\ - g(T^\nabla(Y,X),Z) - g(T^\nabla(X,Z),Y) + g(T^\nabla(Z,Y),X).\end{aligned}$$

Observing that

$$g(J^\nabla_X Y, Z) = g(T^\nabla(Y,Z),X),$$

the theorem follows. □

Of course, the torsion T^∇ and its dual J^∇ depend on the connection ∇. This gives us a condition under which a metric connection will be uniquely defined which we will use often in order to give axiomatic descriptions of connections.

Corollary 2.1.8. *Any metric connection $\nabla = \nabla^g + A^\nabla$ is uniquely defined by an expression for A^∇ independent from the connection itself.*

2.2 Adapted Connections on foliations

In this section we introduce the notion of an adapted connection to a foliation, and consider a number of well-known examples.

Definition 2.2.1. Let $(\mathbb{M}, \mathcal{F}, g)$ be a foliation with vertical distribution $\mathcal{V}$ and a choice of transverse distribution $\mathcal{H}$. We say that a connection ∇ is adapted to the foliation if the connection preserves the foliation; that is if for any vector field $X \in \Gamma(T\mathbb{M})$ it holds that

- for all $Y \in \Gamma(\mathcal{H})$, $\nabla_X Y \in \Gamma(\mathcal{H})$, and
- for all $Z \in \Gamma(\mathcal{V})$, $\nabla_X Z \in \Gamma(\mathcal{V})$.

Generally, the Levi-Civita connection ∇^g is not an adapted connection (the exception being of course a trivial foliation of $\mathbb{R}^n$). We shall usually work with metric connections; we note that it follows from the uniqueness of the Levi-Civita connection as metric and torsion-free that adapted metric connections must have torsion.

2.2.1 Bott's Connection

Standard in the literature on foliations [110] [68] [26] is the notion of the Bott connection, which is a metric connection well-adapted to the splitting.

Theorem 2.2.2 (Generalized Bott Connection)**.** *For* $(\mathbb{M}, g, \mathcal{F})$ *be a foliation with orthogonally splitting metric* $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$*. Then there exists a unique connection* ∇^B *over* $\mathbb{M}$ *called the Bott connection satisfying the following:*

1. ∇^B *is metric,*
2. ∇^B *respects the foliation,*
3. $T^B(\mathrm{E}, \mathrm{E}) \subseteq \mathrm{E}^\perp$,
4. *For* $Z \in \mathrm{E}^\perp$*, the tensor* $C_Z \in \mathrm{E} \otimes \mathrm{E}^* \otimes \mathrm{E}^*$ *given by* $C_Z(X, Y) = g(T^B(X, Z), Y)$ *is symmetric.*

where properties 3 and 4 hold for both $\mathrm{E} = \mathcal{H}$ *and* $\mathrm{E} = \mathcal{V}$*.*

Proof. We will show that the four properties of the Bott connection uniquely determine it. Suppose that ∇ is a connection satisfying properties 1-4. From corollary 2.1.8 it is sufficient to prove that

$$A^{\nabla}(X,Y) = \frac{1}{2}\left(T^{\nabla}(X,Y) - J^{\nabla}_{Y}X - J^{\nabla}_{X}Y\right)$$

can be determined independently of ∇. We proceed by cases. Observe that $g(\nabla_X Y, Z) = 0$ whenever $Y \in \mathrm{E}, Z \in \mathrm{E}^{\perp}$ by property 2, so we only consider $g(A(X,Y),Z)$ for the case $Y, Z \in \mathrm{E}$. If $X \in \mathrm{E}$ we see from property 3 that

$$g(T(X,Y),Z) = g(J_Y X, Z) = g(J_X Y, Z) = 0$$

so $g(A^{\nabla}(E,E),E) = 0$.

If $X \in \mathrm{E}^{\perp}$, it follows from property 2 that

$$g(T(Y,Z),X) = -g([Y,Z],X)$$

or equivalently

$$J_X Y = -(\mathcal{L}^{\flat}_{Y}(\cdot, X))^{\sharp}$$

and from property 4 that

$$g(T(X,Y) - J_Y X, Z) = 0.$$

and so A^{∇} is uniquely determined by properties 1-4.

$\square$

Remark 2.2.3. Notice that all 4 properties are necessary to determine A, which shows that they constitute a minimal axiomatic description of the Bott connection.

Corollary 2.2.4. *The Bott connection can be written explicitly as*

$$\nabla^B_X Y = \begin{cases} \operatorname{pr}_{\mathcal{H}} \nabla^g_X Y & X, Y \in \mathcal{H} \\ \operatorname{pr}_{\mathcal{H}}[X,Y] + \hat{A}_X Y & X \in \mathcal{V}, Y \in \mathcal{H} \\ \operatorname{pr}_{\mathcal{V}}[X,Y] + \hat{A}_X Y & X \in \mathcal{H}, Y \in \mathcal{V} \\ \operatorname{pr}_{\mathcal{V}} \nabla^g_X Y & X, Y \in \mathcal{V} \end{cases}$$

where ∇^g denotes the Levi-Civita connection, and the (1,2)-tensor $\hat{A}$ is given by

$$\hat{A}_X Y = \frac{1}{2} \left((\mathcal{L}_{\operatorname{pr}_{\mathcal{V}} X}\, g)(\operatorname{pr}_{\mathcal{H}} Y, \operatorname{pr}_{\mathcal{H}} \cdot) + (\mathcal{L}_{\operatorname{pr}_{\mathcal{H}} X}\, g)(\operatorname{pr}_{\mathcal{V}} Y, \operatorname{pr}_{\mathcal{V}} \cdot) \right)^{\sharp}.$$

Its torsion has the form

$$T^B(X,Y) = -\operatorname{pr}_{\mathcal{V}}[\operatorname{pr}_{\mathcal{H}} X, \operatorname{pr}_{\mathcal{H}} Y] + \hat{A}_X Y - \hat{A}_Y X.$$

Proof. This follows directly from the explicit expression for A obtained in the proof of the previous theorem, observing that for $Y, Z \in \mathrm{E}, X \in \mathrm{E}^{\perp}$

$$2g(A^{\nabla^B}(X,Y), Z) = g([Y,Z], X) = (\mathcal{L}_Y\, g)(Z, X) - g(Z, [Y,X])$$

and so

$$\begin{aligned} g(\nabla^B_X Y, Z) &= g(\nabla^g_X Y + A^{\nabla^B}(X,Y), Z) \\ &= g([X,Y], Z) + \hat{A}_X Y \end{aligned}$$

after an application of Koszul's formula. □

Remark 2.2.5. Notice, our expression for the Bott connection is symmetric in $\mathcal{H}$ and $\mathcal{V}$ except for one term in the torsion; this only occurs because the bracket of vertical fields remains vertical, as can be seen by considering basic fields (see lemma 4.3.3). In some sense then, the connection doesn't see the difference between the distributions.

Frequently [110, 115] the name 'Bott connection' is ascribed to the restricted case of totally geodesic foliations with bundle-like metric, or to the partial connection along the leaves $\mathcal{V}$. In this setting $\hat{A}$ vanishes and the connection and torsion simplify for $X \in \mathrm{E}$ to

$$\nabla^B_X Y = \begin{cases} \mathrm{pr}_E \nabla^g_X Y & Y \in \mathrm{E} \\ \mathrm{pr}_{E^\perp}[X, Y] & Y \in \mathrm{E}^\perp \end{cases}$$

and

$$T^B(X, Y) = -\,\mathrm{pr}_{\mathcal{V}}[\mathrm{pr}_{\mathcal{H}} X, \mathrm{pr}_{\mathcal{H}} Y]$$

2.2.2 Tanaka-Webster-Tanno Connection

Contact Manifolds

Definition 2.2.6. We call $(\mathbb{M}, \eta)$ a contact manifold if $\mathbb{M}$ is a $2n + 1$ dimensional manifold and η is a 1-form such that $\eta \wedge (d\eta)^n$ is a volume form on $\mathbb{M}$.

Proposition 2.2.7. *Let* $(\mathbb{M}, \eta)$ *be a contact manifold. There exist on* $\mathbb{M}$ *a unique vector field* ξ*, a Riemannian metric* g*, and a* $(1, 1)$*-tensor field* $\bar{J}$ *such that for all* $X, Y \in \Gamma(T\mathbb{M})$

1. $\eta(\xi) = 1$, $\iota_\xi d\eta = 0$,

2. $g(X, \xi) = \eta(X)$,

3. $g(X, \bar{J}Y) = d\eta(X, Y)$,

4. $\bar{J}^2 X = -X + \eta(X)\xi$.

See [11] for a proof of the proposition, as well as an introduction to contact manifolds.

ξ is called the Reeb vector field, and such a metric is said to be compatible with the contact structure. A contact manifold $(\mathbb{M}, \eta)$ can be canonically equipped with a codimension one foliation $\mathcal{F}_\xi$ by choosing the horizontal distribution to be $\mathcal{H} = \ker \eta$ and the vertical distribution $\mathcal{V}$ to be generated by the Reeb vector field ξ . This is known as the Reeb foliation.

Lemma 2.2.8. *The characteristic foliation $\mathcal{F}_\xi$ is totally-geodesic.*

Proof. This is equivalent to requirement $\iota_\xi d\eta = 0$, see [40, lemma 6.3.3]. □

Remark 2.2.9. In property 3 we take the modern convention, but in the original work by Tanno [108] he writes instead

$$2g(X, \bar{J}Y) = d\eta(X, Y).$$

This is just a choice of normalization, but it will affect the H-type property that we introduce in chapter 3; see remark 3.2.3.

Remark 2.2.10. There has been established a large collection of statements ranging from much weaker to much stronger on contact manifolds and related structures. See [40] for a complete picture.

Theorem 2.2.11 (Tanno [108]). *Let $(\mathbb{M}, \eta)$ be a contact manifold. There exists a unique connection ∇^T on $T\mathbb{M}$ satisfying*

1. *∇^T is metric,*

2. *$\nabla^T \xi = 0$,*

3. *$T^T(X, Y) = d\eta(X, Y)\xi$ for any $X, Y \in \mathcal{H}$,*

4. *$T^T(\xi, \bar{J}Y) = -\bar{J}T^T(\xi, Y)$ for any $Y \in T\mathbb{M}$*

5. *$\nabla^T_X \bar{J} = Q(\cdot, X)$ for any $X, Y \in T\mathbb{M}$,*

where the Tanno tensor Q is the $(1, 2)$-tensor field determined by

$$Q(Y, X) = (\nabla^g_X \bar{J})Y + \big((\nabla^g_X \eta)\bar{J}Y\big)\, \xi + \eta(Y)\bar{J}(\nabla^g_X \xi).$$

This connection is known as Tanno's connection, or sometimes as the Tanaka-Webster-Tanno connection.

Proof. Let ∇ be a connection obeying the properties above; by corollary 2.1.8 it is enough to find an expression for

$$A(X, Y) = \frac{1}{2}\left(T(X, Y) - J_X Y - J_Y X\right)$$

independent of the connection. We begin by proving two lemmas.

Lemma 2.2.12. *Any connection satisfying properties 1 and 2 must respect the foliation.*

Proof. Notice first, for $X \in \mathcal{V}$

$$\nabla_Y X = (Y \cdot \eta(X))\xi \in \mathcal{V}.$$

For $X \in \mathcal{H}$,

$$g(\nabla_Y X, \xi) = -(\nabla_Y g)(X, \xi) + Y \cdot g(X, \xi) - g(X, \nabla_Y \xi) = 0$$

so $\nabla_Y X \in \mathcal{H}$, completing the lemma. □

From this, it is clear that we only need to establish expressions for $g(A(X, Y), Z)$ for $Y, Z \in \mathrm{E}$.

Lemma 2.2.13. *For any connection obeying properties 1, 3, and 4 it will hold that*

- $T(\mathcal{H}, \mathcal{H}) \in \mathcal{V}$
- $T(\mathcal{V}, \mathcal{H}) \in \mathcal{H}$
- $T(\mathcal{V}, \mathcal{V}) = 0$

Proof. Observe that if $X, Y \in \mathcal{H}$ that property 3 implies $T(X, Y) = d\eta(X, Y)\xi \in \mathcal{V}$.

We see that the statement $X = -\bar{J}^2 X$ is equivalent to $X \in \mathcal{H}$, and so applying property 4 twice it follows that

$$T(\xi, X) = -T(\xi, \bar{J}^2 X) = -\bar{J}^2 T(\xi, X) \in \mathcal{H}.$$

Finally, for $X, Y \in \mathcal{V}$ we have

$$T(X, Y) = \eta(X)\eta(Y)T(\xi, \xi) = 0$$

from the skew-symmetry of T. □

We now split the proof by cases. For $X, Y, Z \in \mathrm{E}$ or $X \in \mathcal{H}, Y, Z \in \mathcal{V}$, we can apply the lemma and have simply

$$2g(A(X,Y),Z) = g(T(X,Y),Z) - g(T(Y,Z),X) - g(T(X,Z),Y) = 0$$

In the case $X \in \mathcal{V}, Y, Z \in \mathcal{H}$ we begin by considering property 4 of the torsion and find

$$\begin{aligned} T(\xi,Y) &= \bar{J}T(\xi,\bar{J}Y) \\ \nabla_\xi Y - [\xi,Y] &= \bar{J}(\nabla_\xi(\bar{J}Y) - [\xi,\bar{J}Y]) \\ \nabla_\xi Y &= \bar{J}((\nabla_\xi \bar{J})Y + \bar{J}(\nabla_\xi Y) - \bar{J}[\xi,\bar{J}Y] - [\xi,Y] \\ 2\nabla_\xi Y &= \bar{J}Q(Y,\xi) - \bar{J}[\xi,\bar{J}Y] + [\xi,Y] \end{aligned}$$

where we've used property 2 to eliminate several terms, and the Q tensor appears. Since this is an expression for $\nabla_{\mathcal{V}}\mathcal{H}$ independent of the connection we are done. □

Remark 2.2.14. The last case $X \in \mathcal{V}, Y \in \mathcal{H}$ is somehow singular; taking a direct approach to computing $A(X,Y)$ gives only a formula for $\nabla^g_X Y$. We note that this case is the only one for which property 5 is essential.

Corollary 2.2.15. *Tanno's connection can be explicitly written as*

$$\nabla^T_X Y = \nabla^g_X Y - \eta(X)\bar{J}Y - \eta(Y)\nabla^g_X \xi + ((\nabla^g_X \eta)Y)\xi$$

or equivalently

$$\nabla^T_X Y = \begin{cases} \mathrm{pr}_{\mathcal{H}} \nabla^g_X Y & X, Y \in \mathcal{H} \\ \mathrm{pr}_{\mathcal{H}} \nabla^g_X Y - \eta(X)\bar{J}Y & X \in \mathcal{V}, Y \in \mathcal{H} \\ \mathrm{pr}_{\mathcal{V}}[X, Y] & X \in \mathcal{H}, Y \in \mathcal{V} \\ \mathrm{pr}_{\mathcal{V}} \nabla^g_X Y & X, Y \in \mathcal{V} \end{cases}$$

Proof. By direct computation, the connection defined by this formula satisfies all of the defining conditions; it follows from uniqueness that this must be Tanno's connection. One important note for the computation is that we can write

$$d\eta(X, Y) = (\nabla^g_X \eta)Y - (\nabla^g_Y \eta)X.$$

Since both $\iota_\xi d\eta = 0$ and $\nabla^g_\xi \eta = 0$ hold, it follows that the expression $(\nabla^g_X \eta)Y$ will vanish unless we have both $X, Y \in \mathcal{H}$. □

The following proposition is often included as part of the definition of Tanno's connection.

Proposition 2.2.16. *η is ∇^T-parallel*

Proof. Observe that for any vector fields X, Y,

$$\begin{aligned} (\nabla^T \eta)(X, Y) &= (\nabla^T_X \eta)Y \\ &= X \cdot \eta(Y) - \eta(\nabla^T_X Y) \\ &= X \cdot \eta(Y) - \eta(X \cdot \eta(Y)\xi - \nabla^T_X \mathrm{pr}_{\mathcal{H}} Y) = 0 \end{aligned}$$

where we use that $\nabla^T \xi = 0$, and that both $\mathcal{H}$ and $\mathcal{V}$ are ∇^T-parallel. □

Remark 2.2.17. A case of particular interest is when $Q \equiv 0$; this condition implies $\bar{J}$ is ∇^T-parallel, and is equivalent to (M, η, J) being a strongly pseudoconvex CR manifold. Moreover, ξ will be a Killing field, and the foliation will be totally geodesic with bundle-like metric. We will investigate this in a later section.

Remark 2.2.18. In a recent work [92] Nagase and Sasaki address this same deficiency of Tanno's connection for computations on contact manifolds; that is, that the $\bar{J}$ map is not parallel. In particular, they define the <u>hermitian Tanno's connection</u>

$$\nabla^{HT}_X Y = \nabla^T_X Y - \frac{1}{2}\bar{J}Q(Y, X).$$

We note that they have taken the normalization $g(X, JY) = d\eta(X, Y)$ (see remark 2.2.9). It can be straightforwardly verified that $\nabla^{HY}\bar{J} = 0$. This simplifies the computation of some curvature quantities of interest, but it's torsion does not have desired symmetry properties for our purposes.

K-contact manifolds

By insisting the the Reeb field is Killing, we can make the canonical foliation $\mathcal{F}_\xi$ have bundle-like metric, at which point the Tanno and Bott connections agree.

Definition 2.2.19. Let $(\mathbb{M}, \eta, g)$ be a contact manifold with compatible metric g. We call $\mathbb{M}$ a K-contact manifold if the associated Reeb field ξ is a Killing field, that is if

$$\mathcal{L}_\xi\, g = 0.$$

Our interest in K-contact manifolds is motivated by the following

Proposition 2.2.20. *Let* $(\mathbb{M}, \eta, g, \mathcal{F}_\xi)$ *be a contact manifold equipped with Reeb foliation* $\mathcal{F}_\xi$ *and compatible metric* g*. Then the following are equivalent:*

1. $(\mathbb{M}, \eta, g)$ *is a* K*-contact manifold,*

2. $(\mathbb{M}, \mathcal{F}_\xi, g)$ *is a totally-geodesic foliation with bundle-like metric* g*.*

Moreover, if the above statements hold then the Bott connection ∇^B *on* $(\mathbb{M}, g, \mathcal{F}_\xi)$ *and Tanno's connection* ∇^T *on* $(\mathbb{M}, \eta, g)$ *coincide.*

Proof. We know from lemma 2.2.8 that $\mathcal{F}_\xi$ is totally-geodesic; since ξ generates $\mathcal{V}$, it is clear that the manifold being K-contact is equivalent to it having bundle-like metric.

The equivalence of ∇^B and ∇^T will now follow, since ∇^T verifies property 1 defining the Bott connection by definition, properties 2 and 3 are precisely lemma 2.2.12 and lemma 2.2.13, respectively, and property 4 defining the Bott connection is verified by a consideration of the torsion property 4 of Tanno's connection using that the metric is bundle-like. In particular, $T(\xi, JY) = \frac{1}{2}(\mathcal{L}_\xi\, g)(\cdot, JY)^\sharp = 0$. □

CR Manifolds

We give a brief collection of definitions; for a full review see [55]. A CR-manifold of type (n,m) is a pair $(\mathbb{M}, T_{1,0}(\mathbb{M}))$ where

- M is a real $(2n+m)$-dimensional smooth manifold,
- $T_{1,0}(\mathbb{M})$ is a complex subbundle of the complexified tangent bundle $T\mathbb{M} \otimes \mathbb{C}$ with complex rank n such that

$$T_{1,0}(\mathbb{M}) \cap T_{0,1}(\mathbb{M}) = \emptyset,$$

where $T_{0,1}(\mathbb{M}) = \overline{T_{1,0}(\mathbb{M})}$ and

- for any open set $U \subseteq \mathbb{M}$,

$$[\Gamma(U, T_{1,0}(\mathbb{M})), \Gamma(U, T_{1,0}(\mathbb{M}))] \subseteq \Gamma(U, T_{1,0}(\mathbb{M})).$$

We then define the horizontal distribution (known as the Levi distribution) on $\mathbb{M}$ as the rank $2n$ subbundle of $T\mathbb{M}$

$$\mathcal{H} = \Re(T_{1,0}(\mathbb{M}) \oplus T_{0,1}(\mathbb{M})).$$

There is a canonical complex structure J_b on $\mathcal{H}$ given by

$$J_b(V + \overline{V}) = i(V - \overline{V}).$$

We can understand J_b in this setting as distinguishing between $T_{1,0}\mathbb{M}$ and $T_{0,1}\mathbb{M}$. In particular, $J_b \colon T_{1,0}\mathbb{M} \to T_{0,1}\mathbb{M}$ and $J_b \colon T_{0,1}\mathbb{M} \to T_{1,0}\mathbb{M}$.

For orientable CR-manifolds of type $(n, 1)$ (which we will refer to just as CR-manifolds for the remainder of the paper) there is a notion of pseudo-Hermitian stucture, which is a globally nonvanishing 1-form θ such that

$$\ker(\theta) \supseteq \mathcal{H}$$

Given a pseudo-Hermitian structure θ we can further define the Levi form

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W})$$

for $Z, W \in T_{1,0}(\mathbb{M})$, and a bilinear structure G_θ

$$G_\theta(X, Y) = (d\theta)(X, J_b Y)$$

for $X, Y \in \mathcal{H}$. Notice that L_θ and (the $\mathbb{C}$-bilinear $\mathcal{H} \otimes \mathbb{C}$ extension of) G_θ agree on $T_{1,0}(\mathbb{M}) \otimes T_{0,1}(\mathbb{M})$. Moreover,

$$G_\theta(J_b X, J_b Y) = G_\theta(X, Y).$$

Remark 2.2.21. This will imply that CR manifolds obey are H-type foliations, see definition 3.2.2.

There is a unique nonwhere vanishing tangent vector field ξ on $T(\mathbb{M})$ such that

$$\theta(\xi) = 1, \quad \iota_\xi d\theta = 0$$

which we call the characteristic direction of $(\mathbb{M}, \theta)$, and define the vertical distribution $\mathcal{V} = \mathbb{R}\xi$ to be the subbundle of $T\mathbb{M}$ generated by ξ. We see that

$$T\mathbb{M} = \mathcal{H} \oplus \mathcal{V}.$$

On a nondegenerate CR-manifold (that is, equipped with a pseudo-Hermitian structure θ with nondegenerate associated Levi form) there is a canonical semi-

Riemannian metric g_θ referred to as the Webster metric given by

$$g_\theta(X,Y) = \begin{cases} G_\theta(X,Y) & X,Y \in \mathcal{H} \\ 0 & X \in \mathcal{H}, Y \in \mathcal{V} \\ \theta(X)\theta(Y) & X,Y \in \mathcal{V} \end{cases}$$

The signature (r,s) of L_θ is constant, and the signature of g_θ is always $(2r+1, 2s)$. Thus if L_θ is positive definite we see that g_θ is Riemannian.

We begin with a convenient definition.

Definition 2.2.22. Let T^∇ be the torsion of a linear connection ∇ on a CR manifold $(\mathbb{M}, \theta)$ with characteristic direction ξ. We say that T^∇ is pure if

1. $T^\nabla(Z,W) = 0$,
2. $T^\nabla(Z,\overline{W}) = 2iL_\theta(Z,\overline{W})\xi$, and
3. $\tau \circ J_b + J_b \circ \tau = 0$

for any $Z, W \in T_{1,0}(\mathbb{M})$, where the pseudo-Hermitian torsion τ is the endomorphism of $T\mathbb{M}$ given by

$$\tau(X) = T^\nabla(\xi, X).$$

In this setting, there exists a well known adapted connection.

Theorem 2.2.23 (Tanaka [107], Webster [114])**.** *Let* $(\mathbb{M}, T_{1,0}(\mathbb{M}))$ *be a nondegenerate strongly pseudoconvex CR manifold and* θ *a pseudo-Hermitian structure on* $\mathbb{M}$*. Let* ξ *be the characteristic direction of* $(\mathbb{M},\theta)$*,* J_b *the complex structure on* $T\mathbb{M}$*, and* g_θ *be the Webster metric of* (M,θ)*. There is a unique metric connection* ∇^{TW} *on* $\mathbb{M}$ *satisfying*

1. *$\mathcal{H}$ is ∇^{TW}-parallel,*

2. *$\nabla^{TW} J_b = 0$, and*

3. *the torsion $T^{\nabla^{TW}}$ is pure.*

We call ∇^{TW} the Tanaka-Webster connection, which was introduced independently by Tanaka in [107] and Webster in [114]. A thorough discussion can be found in [55].

Proposition 2.2.24. *The Tanaka-Webster connection on $(\mathbb{M}, \theta)$ coincides with the Bott connection on $(\mathbb{M}, \mathcal{H}, g_\theta)$.*

Proof. We see that a nondegenerate CR-manifold is a contact manifold, identifying the characteristic direction ξ with the Reeb field ξ and

$$\theta = \eta$$

$$g_\theta = d\eta$$

$$J_b = \bar{J}|_{\mathcal{H}}$$

In particular, Tanno's Q tensor vanishes, and as a consequence the manifold is K-contact and the defining properties of the Tanaka-Webster connection are precisely those of Tanno's connection. We conclude that in this setting that the Bott, Tanno's and Tanaka-Webster connection coincide.

□

Remark 2.2.25. From this, we can consider Tanno's connection a generalization of the Tanaka-Webster connection. In fact, this is exactly the point of view that Tanno took in his original paper.

2.2.3 Biquard's Connection

Quaternion Contact Manifolds

Definition 2.2.26. Let $(\mathbb{M}, g, \mathcal{H})$ be a $4n+3$ dimensional Riemannian manifold with a codimension 3 distribution $\mathcal{H}$ such that

1. $\mathcal{H}$ has a $Sp(n)Sp(1)$-structure; that is there exists a rank 3 bundle $\mathcal{Q}$ consisting of $(1,1)$-tensors on $\mathcal{H}$ locally generated by three almost-complex structures I_1, I_2, I_3 on $\mathcal{H}$ satisfying the quaternion relations $I_1 I_2 I_3 = -id$ which are hermitian compatible with the metric, that is

$$g(I_j \cdot, I_j \cdot) = g(\cdot, \cdot)$$

   for $j \in \{1, 2, 3\}$.

2. $\mathcal{H}$ is locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ such that

$$g(I_j X, Y) = d\eta_j(X, Y)$$

   for $j \in \{1, 2, 3\}$.

We then call $(\mathbb{M}, g, \mathcal{H}, \mathcal{Q})$ a quaternionic contact manifold or qc manifold.

There is an appropriate generalization of the Reeb field to the qc manifold case:

Definition 2.2.27. Suppose there exists a supplementary subspace $\mathcal{V}$ to $\mathcal{H}$ and an orthonormal basis $\{\xi_1, \xi_2, \xi_3\}$ for $\mathcal{V}$ such that

1. $\eta_i(\xi_j) = \delta_{ij}$;

2. $(\iota_{\xi_j} d\eta_j)_{\mathcal{H}} = 0$; and

3. $(\iota_{\xi_j} d\eta_k)_{\mathcal{H}} = -(\iota_{\xi_k} d\eta_j)_{\mathcal{H}}$.

The fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields, in keeping with the nomenclature for contact manifolds.

Remark 2.2.28. Biquard [36] showed that Reeb vector fields always exist for a qc manifold of dimension $4n + 3 > 7$.

Observe that we can define a map $\varphi\colon \mathcal{V} \to \mathrm{End}(\mathcal{H})$ by

$$\sum a_i \xi_i \mapsto \sum a_i I_i$$

In this setting, we have a canonical reference connection:

Theorem 2.2.29 (Biquard [36])**.** *Let $(\mathbb{M}, g, \mathcal{H}, \mathcal{Q})$ be a quaternionic contact manifold equipped with Reeb fields $\{\xi_i\}$ forming a basis for $\mathcal{V} = \mathcal{H}^{\perp}$. Then there exists a unique connection ∇^{Bi} with torsion T^{Bi} on $\mathbb{M}$.*

1. *∇^{Bi} is metric;*

2. *∇^{Bi} respects the splitting $\mathcal{H} \oplus \mathcal{V}$;*

3. *$\nabla^{Bi}\varphi = 0$;*

4. *$T^{Bi}(\mathcal{H}, \mathcal{H}) \subseteq \mathcal{V}$;*

5. *for $X \in \mathcal{V}$, the operator $T^{Bi}_X(\cdot) := T^{Bi}(X, \cdot)\colon \mathcal{H} \to \mathcal{H}$ is in $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^{\perp} \subset \mathfrak{gl}(4n)$.*

The connection ∇^{Bi} is called the Biquard connection on $(\mathbb{M}, g, \mathcal{H}, \mathcal{Q})$.

Remark 2.2.30. Duchemin [56] showed that assuming the existence of a triple of Reeb fields, the Biquard connection is well defined for a 7-dimensional qc manifold.

Corollary 2.2.31. *The Biquard connection on* $(\mathbb{M}, g, \mathcal{H}, \mathcal{Q})$ *and the Bott connection on* $(\mathbb{M}, \mathcal{H}, g)$ *coincide.*

Proof. The proof follows from verifying the defining properties of the Bott connection. In particular, property 5 will imply the torsion symmetry. See [12, Section 1.2] for more details. □

2.2.4 Hladky's Connection

We begin in the general setting of sub-Riemannian manifolds with metric complement on which there exists a connection known as Hladky's connection. This connection generalizes the Bott connection.

Graded Sub-Riemannian Manifolds with Compatible Metric

Definition 2.2.32. We call a sub-Riemannian manifold $(\mathbb{M}, g_{\mathcal{H}}, \mathcal{H})$ equipped with a choice of supplementary distribution $\mathcal{V}$ a sub-Riemannian manifold with complement or sRC manifold.

We say that a sRC manifold $(\mathbb{M}, g, \mathcal{H}, \mathcal{V})$ is r-graded if there are smooth constant rank bundles $\mathcal{V}^{(j)}, 0 < j \leq r$, such that

$$\mathcal{V} = \mathcal{V}^{(1)} \oplus \cdots \oplus \mathcal{V}^{(r)}$$

and

$$\mathcal{H} \oplus \mathcal{V}^{(j)} \oplus [\mathcal{H}, \mathcal{V}^{(j)}] \subseteq \mathcal{H} \oplus \mathcal{V}^{(j)} \oplus \mathcal{V}^{(j+1)}$$

for all $0 \leq j \leq r$ with the convention that $\mathcal{V}^{(0)} = \mathcal{H}$ and $\mathcal{V}^{(j)} = 0$ for $j < 0$ and $r < j$.

An adapted metric extension for an r-graded sRC manifold $(\mathbb{M}, g_{\mathcal{H}}, \mathcal{V}, \mathcal{H})$ is a Riemannian metric g that agrees with $g_{\mathcal{H}}$ on $\mathcal{H}$ and makes the split

$$T\mathbb{M} = \mathcal{H} \bigoplus_{1 \leq j \leq r} \mathcal{V}^{(j)}$$

orthogonal.

For convenience, we shall denote by $X^{(j)}$ a section of $\mathcal{V}^{(j)}$ and set

$$\hat{\mathcal{V}}^{(j)} = \bigoplus_{k \neq j} \mathcal{V}^{(k)}$$

Theorem 2.2.33 (Hladky [68]). *Let* $(\mathbb{M}, g, \mathcal{H}, \mathcal{V})$ *be an r-graded sRC manifold with adapted metric extension* g. *There exists a unique connection* $\nabla^{Hl^{(r)}}$ *with torsion* $T^{Hl^{(r)}}$ *such that*

1. $\nabla^{Hl^{(r)}}$ *is metric, that is* $\nabla^{Hl^{(r)}} g = 0$;
2. $\mathcal{V}^{(j)}$ *is parallel for all* j;
3. $T^{Hl^{(r)}}(\mathcal{V}^{(j)}, \mathcal{V}^{(j)}) \subseteq \hat{\mathcal{V}}^{(j)}$ *for all* j;
4. $g(T^{Hl^{(r)}}(X^{(j)}, Y^{(k)}), Z^{(j)}) = g(T^{Hl^{(r)}}(Z^{(j)}, Y^{(k)}), X^{(j)})$ *for all* j, k.

Proof. Suppose that ∇ is a connection satisfying properties 1-4. Since ∇ is metric, we have the Koszul relation

$$2g(\nabla_X Y, Z) = 2g(\nabla^g_X Y, Z) + g(T(X,Y), Z) - g(T(Y,Z), X) + g(T(Z,X), Y)$$

Because ∇ is parallel for each $\mathcal{V}^{(j)}$, we need only consider the cases $Y, Z \in \mathcal{V}^{(j)}$. If $X \in \mathcal{V}^{(j)}$ we see that

$$g(T(X,Y),Z) = g(T(Y,Z),X) = g(T(Z,X),Y) = 0$$

so $\nabla_X Y = \mathrm{pr}_j \nabla^g_X Y$.

On the other hand, if $X \in \hat{\mathcal{V}}^{(j)}$ we have that

$$g(T(Y,Z),X) = g([Z,Y],X) \quad \text{and} \quad g(T(X,Y),Z) + g(T(Z,X),Y) = 0$$

so we can conclude

$$2g(\nabla_X Y, Z) = 2g(\nabla^g_X Y, Z) + g([Z,Y],X).$$

We thus have expressions for $\nabla_X Y$ independent of ∇, and so if a connection satisfies properties 1-4 it must be unique.

From the expression derived above, we can write

$$\nabla^{(r)}_X Y = \begin{cases} \mathrm{pr}_i \nabla^g_X Y & X, Y \in \mathcal{V}^{(i)} \\ \mathrm{pr}_i [X,Y] + A^i_X Y & Y \in \mathcal{V}^{(i)}, X \in \hat{\mathcal{V}}^{(i)} \end{cases}$$

where the tensor $A^i_X \in T^*\mathbb{M} \otimes T\mathbb{M}$ is given by

$$2A^i_X Y = \sharp\left((\mathcal{L}_{\mathrm{pr}_i X}\, g(\mathrm{pr}_{\hat{\mathcal{V}}^{(i)}} Y, \mathrm{pr}_{\hat{\mathcal{V}}^{(i)}} \cdot) + (\mathcal{L}_{\mathrm{pr}_{\hat{\mathcal{V}}^{(i)}} X}\, g)(\mathrm{pr}_i Y, \mathrm{pr}_i \cdot) \right).$$

The expression can be directly confirmed to satisfy the properties of the Hladky connection, completing the proof. □

Remark 2.2.34. If $X, Y \in \mathcal{H}$ we see that $\nabla^{Hl^{(r)}} X$ and $T^{Hl^{(r)}}(X, Y)$ are independent of the choice of grading and metric extension. Moreover, an r-graded sRC manifold also admits a k-grading (for all $1 \leq k < r$) given by

$$\tilde{\mathcal{V}}^{(j)} = \mathcal{V}^{(j)}, 0 \leq j < k, \qquad \tilde{\mathcal{V}}^{(k)} = \bigoplus_{j \geq k} \mathcal{V}^{(j)}$$

and then associated to each k-grading there is a connection $\nabla^{Hl^{(k)}}$. For this entire family of connections, $\nabla^{Hl^{(j)}} X^{(k)} = \nabla^{Hl^{(r)}} X^{(k)}$ whenever $0 \leq k < j$, so in particular for a horizontal vector field X it holds that

$$\nabla^{Hl^{(1)}} X = \nabla^{Hl^{(2)}} X = \cdots \nabla^{Hl^{(r)}} X$$

and so the differences between the connections $\nabla^{Hl^{(k)}} X$ can be viewed as a choice of how to differentiate vertical vector fields.

Definition 2.2.35. Let $(\mathbb{M}, g, \mathcal{H}, \mathcal{V})$ be an r-graded sRC manifold with extended metric. We will call $\nabla^{Hl} := \nabla^{Hl^{(1)}}$ the Hladky connection.

We see that a foliated manifold with horizontal bundle $\mathcal{H}$ is a 1-graded sRC manifold; we can thus understand the Hladky connection as a generalization of the Bott connection.

Corollary 2.2.36. *On a foliation* $(\mathbb{M}, g, \mathcal{F}, \mathcal{H})$ *the Hladky connection and the Bott connection coincide.*

2.3 Metric Connections with Metric Adjoint

We will now discuss a notion of connection that is not generally adapted to the foliation, but is useful in the computation of geodesics.

Definition 2.3.1. Let $(\mathbb{M}, g)$ be a Riemannian manifold. For a connection ∇ on $\mathbb{M}$, we define its adjoint connection

$$\hat{\nabla}_X Y = \nabla_X Y - T(X,Y) = \nabla_Y X + [X,Y]$$

where $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ is the torsion of ∇.

Observe, $\hat{\hat{\nabla}} = \nabla$ since

$$\hat{\hat{\nabla}}_X Y = \hat{\nabla}_Y X + [X,Y] = \nabla_X Y.$$

For a metric connection ∇, it is not always the case that its adjoint $\hat{\nabla}$ is metric.

Lemma 2.3.2. *The adjoint of a metric connection ∇ is metric if and only if the tensor $T^\flat$ is completely skewsymmetric.*

Proof. Directly,

$$(\hat{\nabla}_Z g)(X,Y) = g(T(Z,X),Y) + g(X,T(Z,Y))$$

which vanishes if and only if

$$g(T(Z,X),Y) = -g(T(Z,Y),X)$$

For all vector fields X, Y, Z. Since the torsion T of any connection is always skew-symmetric, this is equivalent to $T^\flat(X,Y,Z) = g(T(X,Y),Z)$ being completely skewsymmetric. □

We'll see that it's often desirable to have a metric connection with metric adjoint; it is clear that the Levi-Civita connection has metric adjoint because the torsion-free condition implies that it is self-adjoint. In contrast, adapted connections to a foliation do not generally have metric adjoint. In this situation, one option is to use the following result.

Lemma 2.3.3. *Let* $(\mathbb{M}, g)$ *be a Riemannian manifold with a metric connection* ∇*. Define a skew-symmetric tensor* $J_X Y$ *by* $g(J_X Y, Z) = g(Z, T(X,Y))$*. Then the* *<u>associated metric adjoint connection</u>*

$$D_X Y = \nabla_X Y + J_X Y$$

is metric with metric adjoint.

Proof. First, we see that D is metric since

$$(D_X g)(Y,Z) = (\nabla_X g)(Y,Z) - g(J_X Y, Z) - g(Y, J_X Z) = 0.$$

Also, we see that the tensor $T^\flat$ is completely skewsymmetric since

$$\begin{aligned} g(T^D(X,Y),Z) &= g(T(X,Y) + J_X Y - J_Y X, Z) \\ &= -g(J_Z Y + T(Z,Y) - J_Y Z, X) \\ &= -g(T^D(Z,Y),X) \end{aligned}$$

so applying the previous lemma we are finished. □

Lemma 2.3.4. *Let* $(\mathbb{M}, g, \mathcal{F})$ *be a foliation with adapted metric connection* ∇*. The associated connection* $D = \nabla + J$ *will preserve the foliation if any only if* $T^{\nabla}(\mathcal{H}, \mathcal{V}) = 0$.

Proof. For $Y \in E, Z \in E^{\perp}$,

$$g(D_X Y, Z) = -g(X, T^{\nabla}(Y, Z))$$

which vanishes if and only if $T(\mathcal{H}, \mathcal{V}) = 0$. □

Corollary 2.3.5. *On a foliation* $(\mathbb{M}, g, \mathcal{F})$ *the associated metric adjoint connection to the Bott connection preserves the foliation if and only if the foliation is totally geodesic with bundle-like metric.*

2.3.1 The geodesic equation for connections with torsion

In this section, we investigate the variational properties of curves in terms of connections with torsion. We are especially interested in studying the properties of locally length-minimizing curves.

In the following, let $(\mathbb{M}, g)$ be a Riemannian manifold and let ∇ be a g-metric connection on $\mathbb{M}$. Denote by $D = \nabla + J$ the associated metric adjoint connection to ∇, and let $\gamma \colon [0, T] \to \mathbb{M}$ be a smooth curve.

Definition 2.3.6. We say that γ is a geodesic if it is a local length-minimizer. More preciesly, γ is a geodesic if there exists some sufficiently small $0 < t_0 \leq T$ such that for $0 \leq t < t_0$ it will always hold that γ will be the unique shortest curve between $\gamma(0)$ and $\gamma(t)$.

We wish to complement this definition with a definition in terms of connections. Heuristically, we understand that geodesics are curves with constant acceleration, and interpreting connections as a notion of directional derivative leads to the following.

Definition 2.3.7. Let γ be such that its velocity field $\dot{\gamma} = d\gamma\left(\frac{d}{dt}\right)$ has constant magnitude and is D-parallel. That is,

$$g(\dot{\gamma}, \dot{\gamma}) = C$$
$$D_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}} + J_{\dot{\gamma}})\dot{\gamma} = 0.$$

We then call γ a ∇-geodesic.

Remark 2.3.8. In the literature (e.g. [35]) we sometimes have the definition that γ is a ∇-geodesic if it is ∇-parallel. We prefer the above definition so as to guarantee that for any metric connection, ∇-geodesics are geodesics as we will see below in lemma 2.3.10.

It is a well established result in Riemannian geometry that the geodesics of a Riemannaian manifold $(\mathbb{M}, g)$ are precisely the ∇^g-geodesics for the Levi-Civita connection ∇^g. Since it is torsion free, the equation is simply

$$\nabla^g_{\dot{\gamma}}\dot{\gamma} = 0$$

which indicates why this result is often preferred.

Definition 2.3.9. Let $(\mathbb{M}, g)$ be a Riemannian manifold and let $\gamma\colon [0, T] \to \mathbb{M}$ be a smooth curve. If $c\colon [-\varepsilon, \varepsilon] \times [0, T]$ is such that

- $c(s, 0) = \gamma(0)$ for all $s \in [-\varepsilon, \varepsilon]$, and

- $c(0,t) = \gamma(t)$ for all $t \in [0,T]$

then we call c a variation of γ.

One can characterize geodesics as solutions to a variational problem.

Lemma 2.3.10 ([25], Lemma B.1)**.** *Let c be a variation of γ with fixed endpoint $c(s,T) = \gamma(T)$ for all $s \in [-\varepsilon, \varepsilon]$. Then denoting $S = dc\left(\frac{d}{ds}\right)\big|_{s=0}$,*

$$\frac{d}{ds}\bigg|_{s=0} \ell(c(s,\cdot)) = -\int_0^T g\left(D_{\dot\gamma}\dot\gamma, S\right) dt.$$

Corollary 2.3.11. *A curve γ will be a geodesic if and only if it is a ∇-geodesic for some (and therefore any) metric connection ∇.*

2.3.2 Jacobi fields and the comparison principle

We are also interested in the properties of the field $S = dc\left(\frac{d}{ds}\right)$ for a variation of geodesics. See [78] for a complete discussion of established results.

Lemma 2.3.12. *Let $\gamma\colon [0,T] \to \mathbb{M}$ be a geodesic, and let $c(s,t)$ be a geodesic variation of γ in the sense that for any fixed s the map $\gamma_s(t) = c(s,t)$ is a geodesic. Then $S = dc\left(\frac{d}{ds}\right)$ will satisfy the Jacobi equation*

$$D_{\dot\gamma}\hat{D}_{\dot\gamma}S + R(S,\dot\gamma)\dot\gamma = 0$$

where R is the Riemann curvature tensor associated to D.

Proof. See [35, 96]. ∎

In light of this, we define the Jacobi operator

$$\mathcal{Z}(W) = D_{\dot{\gamma}}\hat{D}_{\dot{\gamma}}W + R(W, \dot{\gamma})\dot{\gamma}.$$

We say that a vector field W solving $\mathcal{Z}(W) = 0$ is a Jacobi field. By lemma 2.3.12 the variational field $S = dc\left(\frac{d}{ds}\right)$ associated to a geodesic variation c must always be a Jacobi field; this suggests that we can determine controls on the behavior of geodesics by that of Jacobi fields.

Theorem 2.3.13 (Comparison Theorem)**.** *Let* $(\mathbb{M}, g)$ *be a Riemannian manifold equipped with g-metric connection* ∇*. Let* $x, \bar{x} \in \mathbb{M}$*, and suppose we have the following:*

- *A unit speed geodesic* $\gamma\colon [0,T] \to \mathbb{M}$ *joining* $x = \gamma(0)$ *and* $\bar{x} = \gamma(T)$ *that is length minimizing on the entire interval* $[0,T]$*.*
- *A Jacobi field* V *for* $D = \nabla + J$ *such that* $V(x) = 0$*.*

Defining the distance function $r(y) = d(x,y)$ *(in particular* $r(\gamma(t)) = t$*), it will hold that for any vector field* W *such that* $W|_\gamma \perp \dot{\gamma}$ *and agreeing with* V *at* x *and* $\bar{x}$*,*

$$\mathrm{Hess}^D(r)(V,V) \leq g(W, D_{\dot{\gamma}}W)$$

when both sides are evaluated at $\bar{x}$*, with equality if and only if* $W = V$ *is a Jacobi field.*

Proof. We refer to [25] for the complete result, but the essential idea is to compute the Hessian at $\gamma(T)$ as the integral of a curvature quantity determined by a Jacobi field vanishing at $\gamma(0)$ along γ.

Lemma 2.3.14. *Let* $(\mathbb{M}, g)$ *be a Riemannian manifold equipped with g-metric connection* ∇*. Let* $x, \bar{x} \in \mathbb{M}$*, and suppose* $\gamma\colon [0,T] \to \mathbb{M}$ *is a unit speed geodesic joining* $x = \gamma(0)$ *and* $\bar{x} = \gamma(T)$ *that is length minimizing on the entire interval* $[0,T]$*. Let* $w \in T_{\bar{x}}\mathbb{M}$ *such that* $w \perp \dot{\gamma}(T)$*. Then*

$$\mathrm{Hess}^D(r)(w,w) = \int_0^T \Big(g(D_{\dot{\gamma}}V, \hat{D}_{\gamma}V) - R(V, \dot{\gamma}, \dot{\gamma}, V) \Big)\, dt$$

Where V *is the unique Jacobi field along* γ *for* D *with* $V(x) = 0, V(\bar{x}) = w$*.*

One then shows by a variational argument that the index

$$I(W,W) = \int_0^T \Big(g(D_{\dot{\gamma}}W, \hat{D}_{\gamma}W) - R(W, \dot{\gamma}, \dot{\gamma}, W) \Big)\, dt$$

is minimized by Jacobi fields, and from the theorem follows from the uniqueness of the ODE determining the appropriate Jacobi field. □

Chapter 3

H-type Foliations

Much of the content of this chapter overlaps with a paper coauthored with Baudoin, Grong, and Rizzi in 2018. For the complete proofs of those results we will refer to the original paper [24].

In this chapter we discuss a class of sub-Riemannian manifolds introduced in [24] that are equipped with a Riemannian foliation; these complementary directions determine a Clifford module that has geometric consequences for the sub-Riemannian structure, and it can be shown that these results hold independently of the choice of Riemannian complement. This can be viewed in some sense as a more geometric implementation of the Eulerian approach to sub-Riemannian geometry initiated in [21]. The remainder of the thesis will be dedicated to the study of the sub-Riemannian geometry of these objects.

3.1 Motivation

We begin by examining several coincident ideas that motivate the definition of H-type foliations. In particular, we will look at the H-type groups originally introduced by Kaplan and the complementary notion of Clifford structures that naturally arise in this setting.

3.1.1 H-type Groups and Algebras

In [74] Kaplan introduced a family of two-step nilpotent Lie groups motivated by the study of hypoelliptic Laplacians.

Definition 3.1.1. Suppose $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is a real Lie algebra with Lie bracket $[\cdot,\cdot]$ satisfying

$$[\mathfrak{v},\mathfrak{v}] \subseteq \mathfrak{z}, \quad [\mathfrak{v},\mathfrak{z}] = [\mathfrak{z},\mathfrak{z}] = 0.$$

Suppose moreover there is a scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{G}$. Then defining $J\colon \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ by

$$\langle J_Z X, X'\rangle = \langle Z, [X, X']\rangle$$

the algebra is called H-type if

$$J_Z^2 = -\|Z\|^2 \mathrm{Id}$$

Kaplan explores these spaces further in [75, 76]. In particular one can consider the Clifford algebra $\mathbf{Cl}(\mathfrak{z})$ defined as the tensor algebra $T(\mathfrak{z})$ modulo the relation $x \otimes y + y \otimes x = -2\langle x, y\rangle \mathrm{Id}$. Because of the relation

$$J_{z_1} J_{z_2} + J_{z_2} J_{z_1} = -2\langle z_1, z_2\rangle \mathrm{Id}$$

the universal property of Clifford algebras (see [38, 81] for fundamental notions) implies that the J map can be extended to $J\colon \mathbf{Cl}(\mathfrak{z}) \to \mathrm{End}(\mathfrak{v})$. There is a complete classification of these algebras by dimension, and their properties are well-known.

The Lie groups associated to H-type algebras are natural candidates for the consideration of sub-Riemannian geometry, as the Lie algebra describes the tangent space and we can thereby expect a natural notion of sub-Laplacian. In particular, the simplest nontrivial sub-Riemannian geometry, the Heisenberg group (definition 3.3.1) falls under this category. Further consideration of these spaces has been extensive, see for example [52, 45].

3.1.2 Clifford Structures

There is a natural way in which we can describe the action of a Clifford algebra acting on a Riemannian manifold.

Definition 3.1.2. A rank r Clifford structure on a Riemannian manifold $(\mathbb{M}, g)$ is an oriented rank r Euclidean bundle (E, h) over $\mathbb{M}$ together with a non-vanishing algebra bundle morphism, called a Clifford morphism, $\phi : \mathbf{Cl}(E, h) \to \mathrm{End}(T\mathbb{M})$ which maps E into the bundle of skewsymmetric endomorphisms of $T\mathbb{M}$.

These structures naturally encompass the extension of the Kaplan J map. In [89], Moroianu and Semmelmann completely classify the possible parallel Clifford structures over simply-connected Riemannian manifolds by rank, where such a structure is considered parallel if it is preserved by the Levi-Civita connection (as described in section 2.1.1). In particular, there is a strong relationship between the properties of the Clifford algebra and the Riemann curvature tensor.

3.1.3 Model Spaces for Curved Sub-Riemannian Manifolds

Considering the natural Clifford algebra arising from H-type groups and the well-understood Clifford structures on Riemannian manifolds, it seems a natural progression to consider the possible generalization of Clifford structures to the sub-Riemannian setting in which the horizontal distribution locally models an H-type algebra. This was moreover suggested by [21, Remark 2.25].

In particular, there is an ongoing project in the field of sub-Riemannian geometry [68, 100, 5, 84, 21, 22, 104, 4, 62] exploring appropriate generalizations of curvature. If one hopes to accomplish comparison results (as discussed in section 5.1) it is necessary to establish model spaces of sub-Riemannian geometry analogous to the Euclidean space, sphere, and hyperbolic space of Riemannian geometry. Importantly, it should be the case that there is a unifying theory justifying the appropriateness of these spaces as models for comparison. One possible approach is suggested by the notion of H-type foliations.

3.2 H-Type Foliations

Let $(\mathbb{M}, \mathcal{H}, g)$ be a totally geodesic foliation with adapted bundle-like metric $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$. Denote by ∇ the Bott connection. For each $Z \in \Gamma(T\mathbb{M})$ we define an endomorphism $J_Z \in \mathrm{End}(\Gamma(T\mathbb{M}))$ dual to the Bott torsion

$$J_Z X = T^\flat(\mathrm{pr}_{\mathcal{H}} X, \mathrm{pr}_{\mathcal{H}} \cdot, \mathrm{pr}_{\mathcal{V}} Z)^\sharp$$

or equivalently,

$$g_{\mathcal{H}}(J_Z X, Y) = g_{\mathcal{V}}(Z, T(X, Y))$$

for all $X, Y, Z \in \Gamma(T\mathbb{M})$.

Remark 3.2.1. Notice that the J map introduced here is the same as the J tensor defined in eq. (2.1.1) and which was used for determining the existence and uniqueness of connections; this follows from the expression $T(X,Y) = -\operatorname{pr}_{\mathcal{V}}[\operatorname{pr}_{\mathcal{H}} X, \operatorname{pr}_{\mathcal{H}} Y]$. This will fail to be true for foliations that are not both totally geodesic and have bundle-like metric, which partially motivates including these conditions in our definitions.

With this, we can define a structure generalizing Kaplan's H-type groups, introduced in [74].

Definition 3.2.2. We say that $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation if for every $Z \in \Gamma(\mathcal{V})$ the map J_Z is an isometry; equivalently,

$$g(J_Z X, J_Z Y) = \|Z\|^2 g(X,Y) \tag{3.2.1}$$

for all $X, Y \in \Gamma(T\mathbb{M})$.

Equation (3.2.1) will be called the H-type condition. It is a generalization in the sense that it allows for a notion of his J map on sub-Riemannian manifolds defined by foliations.

Remark 3.2.3. Note, some standard presentations of sub-Riemannian manifolds with natural foliations will not be H-type groups, such as the Hopf fibration with the standard metric on the sphere, as we have instead the property

$$g(J_Z X, J_Z Y) = \lambda \|Z\|^2 g(X,Y)$$

for some fixed $\lambda > 0$. In this case we can renormalize the metric as $g = g_{\mathcal{H}} \oplus \frac{1}{\lambda} g_{\mathcal{V}}$ and thereby recover an H-type foliation.

We further distinguish H-type foliations by the behavior of the Bott torsion under covariant differentiation.

Definition 3.2.4.
- If all horizontal covariant derivatives of the Bott torsion vanish we say $(\mathbb{M}, \mathcal{H}, g)$ has horizontally parallel torsion and we write $\nabla_{\mathcal{H}} T = 0$.
- If all covariant derivatives of the Bott torsion vanish we say $(\mathbb{M}, \mathcal{H}, g)$ has completely parallel torsion and we write $\nabla T = 0$.

To exemplify the importance of these definitions, we note the following lemma that will be used often.

Lemma 3.2.5 ([24], Lemmas 2.6 and 2.7)**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *have horizontally parallel torsion. Then,*

- $(\nabla_X J)_Y = -(\nabla_Y J)_X$, *and*
- $R(X,Y)Z = R_{\mathcal{H}}(X,Y)Z + R_{\mathcal{V}}(X,Y)Z + (\nabla_Z T)(X,Y)$

for all $X, Y, Z \in \Gamma(T\mathbb{M})$, *where we define*

$$R_{\mathcal{H}}(X,Y)Z = R(\operatorname{pr}_{\mathcal{H}} X, \operatorname{pr}_{\mathcal{H}} Y) \operatorname{pr}_{\mathcal{H}} Z$$
$$R_{\mathcal{V}}(X,Y)Z = R(\operatorname{pr}_{\mathcal{V}} X, \operatorname{pr}_{\mathcal{V}} Y) \operatorname{pr}_{\mathcal{V}} Z.$$

Remark 3.2.6. The lemma remains true for totally-geodesic foliations with bundle-like metric even in the absence of the H-type condition.

Proof. The proof of the lemma is an interesting exercise in considering the symmetries of the Riemann curvature tensor. In particular, the second claim is proved by decomposing R into terms for each possible projection of components. Applying the first claim, expanding as $\nabla = \nabla^g + A$, and using the Bianchi identity we find that

$$g(R(X,Y)V,W) - g(R(V,W)X,Y) = g((\nabla_V T)(X,Y),W) - g((\nabla_X T)(V,W),Y)$$

Considering the possible projections, the lemma is proved. We refer to [24] and [22, Lemma A.1] for the details. □

In the following we will denote $J_i = J_{Z_i}$ and $J_{ij} = J_{Z_1}J_{Z_2}$, and $J_{ijk} = J_{Z_i}J_{Z_j}J_{Z_k}$ for succinctness.

Lemma 3.2.7 (Basis Lemma)**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation, and denote* $n = \operatorname{rank}(\mathcal{H})$*. Suppose* $\mathcal{B} = \{X_1, \ldots, X_n\}$ *is an orthonormal basis for* $\mathcal{H}_p$*. Define*

$$\begin{aligned} \operatorname{span}_Z(X) &= \{X, J_Z X\} \\ \operatorname{span}_{Z_1,Z_2}(X) &= \{X, J_1X, J_2X, J_{12}X\} \\ \operatorname{span}_{Z_1,Z_2,Z_3}(X) &= \{X, J_1X, J_2X, J_{12}X, J_3X, J_{13}X, J_{23}X, J_{123}X\}. \end{aligned}$$

We have the following:

1. *For any unit* $Z \in \mathcal{V}_p$ *the set* $\{J_Z X_1, \ldots, J_Z X_n\}$ *is an orthonormal basis for* $\mathcal{H}_p$*.*

2. n *is even, and* $n \geq m + 1$*. Write* $n = 2k$*. For any unit* $Z \in \mathcal{V}_p$ *there exists a subset* $\{X_1, \ldots, X_k\} \subset \mathcal{B}$ *so that the set* $\bigcup_{i=1}^k \operatorname{span}_Z(X_i)$ *is an orthonormal basis of* $\mathcal{H}_p$*.*

3. *If* $m \geq 2$ *then* $n = 4k$. *There exist orthogonal unit* $Z_1, Z_2 \in \mathcal{V}_p$ *and a subset* $\{X_1, \ldots, X_k\} \subset \mathcal{B}$ *such that* $\bigcup_{i=1}^{k} \operatorname{span}_{Z_1,Z_2}(X_i)$ *is an orthonormal basis for* $\mathcal{H}_p$.

4. *If* $m \geq 4$ *then* $n = 8k$. *There exist orthogonal unit* $Z_1, Z_2, Z_3 \in \mathcal{V}_p$ *and a subset* $\{X_1, \ldots, X_k\} \subset \mathcal{B}$ *such that* $\bigcup_{i=1}^{k} \operatorname{span}_{Z_1,Z_2,Z_3}(X_i)$ *is an orthonormal basis for* $\mathcal{H}_p$.

5. *If* $n = m + 1$ *then* $n = 2, 4,$ *or* 8.

Proof. The first claim follows from applying the skew-symmetry of J to see that $g(J_Z X, X) = -g(X, J_Z X) = 0$.

The second claim is established by observing that for any $X, Y \in \mathcal{H}, Z \in \mathcal{V}$, $\operatorname{span}_Z(X)$ is linearly independent and moreover one of the following holds

- $\operatorname{span}_Z(X) = \operatorname{span}_Z(Y)$, or
- $\operatorname{span}_Z(X) \cap \operatorname{span}_Z(Y) = \{0\}$.

For an appropriate choice of $X_i \in \mathcal{B}$ the claim follows.

If $m \geq 2$, fix orthogonal $Z_1, Z_2 \in \mathcal{V}$. Then the set $Span_{Z_1,Z_2}(X)$ is linearly independent and we have an analogous statement the the last case.

If $m \geq 4$ then we can always choose orthogonal $Z_1, Z_2, Z_3 \in \mathcal{V}$ so that for $X \in \mathcal{H}$ the set $\operatorname{span}_{Z_1,Z_2,Z_3}(X)$ is linearly independent and we again have analogous statement.

The final claim follows from the previous three. $\square$

The proof could proceed more elegantly by considering the Clifford algebras $\mathbf{Cl}(\mathcal{V})$. We will take this perspective in the following section.

Remark 3.2.8. For $m = 3$, we note that for orthogonal $Z_1, Z_2, Z_3 \in \mathcal{V}$ it can hold that $J_{12} = J_3$, which is why $m = 3$ does not imply $n = 8k$. When this occurs, we say that the J^2 condition is satisfied; we call the particular case $m = 3$ quaternionic. This will be made precise by definition 3.2.16.

3.2.1 H-type submersions

Definition 3.2.9. Suppose $\pi\colon (\mathbb{M}, g) \to (\mathbb{B}, j)$ is a Riemannian submersion with totally geodesic fibers and $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation with horizontally parallel torsion, where $\mathcal{H}$ is the horizontal space of π. We will call $(\mathbb{M}, \mathcal{H}, g, \pi)$ an H-type submersion.

In fact, the Heisenberg group, the Hopf fibration, and the Anti-de Sitter fibrations (which we define in section 3.3.1) are all H-type submersions. We were able to classify all simply connected H-type submersions in [24, theorem 3.15] with parallel horizontal Clifford structures (defined in section 3.4) by consideration of the analogous classification [89] of Clifford structures on Riemannian manifolds.

To understand this, we recall the notion of curvature constancy:

Definition 3.2.10. For $\rho \in \mathbb{R}$, the ρ-curvature constancy of a Riemannian manifold $(\mathbb{M}, g)$ is the subbundle of $T\mathbb{M}$ given at $p \in \mathbb{M}$ by

$$\mathcal{C}_p(\rho, g) = \{v \in T_p\mathbb{M}\colon R(v, x)y = \rho\Big(g(x, y)v - g(v, y)x\Big) \text{ for all } x, y \in T_p\mathbb{M}\}$$

for the Riemann curvature tensor R associated to the Levi-Civita connection.

Gray [60] defined this as one component of a decomposition of the tangent bundle in terms of the behavior of the Riemann curvature tensor. This space is totally

$\mathbb{M}$	$\mathbb{B}$	Fiber	$\mathbf{rank}(\mathcal{H})$	$\mathbf{rank}(\mathcal{V})$
Twistor space	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^2$	$4k$	2
3-Sasakian	Quaternion-Kähler with positive scalar curvature	$\mathbb{S}^3$	$4k$	3
Quaternion-Sasakian	Product of two quaternion-Kähler with positive scalar curvature	$\mathbb{R}P^3$	$4k$	3
$\frac{\mathbf{Sp}(q^++1)\times\mathbf{Sp}(q^-+1)}{\mathbf{Sp}(q^+)\times\mathbf{Sp}(q^-)\times\mathbf{Sp}(1)}$	$\mathbb{H}P^{q^+}\times\mathbb{H}P^{q^-}$	$\mathbb{S}^3$	$4(q^++q^-)$	3
$\frac{\mathbf{Sp}(k+2)}{\mathbf{Sp}(k)\times\mathbf{Spin}(4)}$	$\frac{\mathbf{Sp}(k+2)}{\mathbf{Sp}(k)\times\mathbf{Sp}(2)}$	$\mathbb{S}^4$	$8k$	4
$\frac{\mathbf{SU}(k+4)}{\mathbf{S}(\mathbf{U}(k)\times\mathbf{Sp}(2)\mathbf{U}(1))}$	$\frac{\mathbf{SU}(k+4)}{\mathbf{S}(\mathbf{U}(k)\times\mathbf{U}(4))}$	$\mathbb{R}P^5$	$8k$	5
$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{Spin}(7)}$	$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{SO}(8)}$	$\mathbb{R}P^7$	$8k$, $k\geq 3$, k odd	7
$\frac{\mathbf{Spin}(k+8)}{\mathbf{SO}(k)\times\mathbf{Spin}(7)}$	$\frac{\mathbf{SO}(k+8)}{\mathbf{SO}(k)\times\mathbf{SO}(8)}$	$\mathbb{S}^7$	$8k$, $k=1$, k even	7
Exceptional cases				
$\frac{F_4}{\mathbf{Spin}(8)}$	$\frac{F_4}{\mathbf{Spin}(9)}=\mathbb{O}P^2$	$\mathbb{S}^8$	16	8
$\frac{E_6}{\mathbf{Spin}(8)\mathbf{U}(1)}$	$\frac{E_6}{\mathbf{Spin}(10)\mathbf{U}(1)}=(\mathbb{C}\otimes\mathbb{O})P^2$	$\mathbb{S}^9$	32	9
$\frac{E_7}{\mathbf{Spin}(11)\mathbf{SU}(2)}$	$\frac{E_7}{\mathbf{Spin}(12)\mathbf{SU}(2)}=(\mathbb{H}\otimes\mathbb{O})P^2$	$\mathbb{S}^{11}$	64	11
$\frac{E_8}{\mathbf{Spin}(15)}$	$\frac{E_8}{\mathbf{Spin}^+(16)}=(\mathbb{O}\otimes\mathbb{O})P^2$	$\mathbb{S}^{15}$	128	15

TABLE 3.2.1: [24, Table 1] H-type submersions with a parallel horizontal Clifford structure and $\kappa > 0$.

geodesic and determines a bundle-like metric under our conditions, and in particular, one recovers the equivalence

Theorem 3.2.11 ([24, Theorem 3.11, Remark 3.12])**.** *Let $(\mathbb{M},\mathcal{H},g)$ be a totally geodesic foliation with bundle like metric and let $K\neq 0$. The following are equivalent:*

- *$\mathcal{V}_p\subset\mathcal{C}_p(K,g)$ for all $p\in\mathbb{M}$.*
- *$(\mathbb{M},\mathcal{H},g_{\mathcal{H}}\oplus\frac{1}{4K}g_{\mathcal{V}})$ is an H-type foliation with parallel horizontal Clifford structure with $\kappa=2K$.*

From this we can arrive at the key result.

Theorem 3.2.12 ([24, Corollary 3.14, Theorem 3.15])**.** *Let* $(\mathbb{B}, j)$ *be an* n*-dimensional Riemannian manifold carrying a rank* $m+1$ *parallel nonflat even Clifford structure in the sense of Moroianu-Semmelmann [89] with* $n \neq 8$*. Then the sphere bundle* $S^m \hookrightarrow \mathbb{M} \to \mathbb{B}$ *is an H-type foliation with parallel horizontal Clifford structure. In particular, restricting to the case of submersions we can conclude that [24, Tables 1 & 2] gives the complete list of H-type submersions with* $\kappa \neq 0$*.*

We refer to [24] for the proofs. The essential observation is that the parallel horizontal Clifford structures we introduce and the parallel Clifford structures of [89] are analogous for this construction as follows from theorem 3.2.11 and as such the classification [89, Theorems 3.6 & 3.7] will give us the result. We reproduce table 3.2.1 giving the classification for $\kappa > 0$ for completeness.

These constitute an important class of H-type foliations, as any foliation is locally a submersion remark 1.1.2 and so local properties of H-type foliations are determined by these examples.

3.2.2 Structure of $\mathbf{Cl}(\mathcal{V}) \to \mathrm{End}(\mathcal{H})$

In this subsection we are interested in examining more closely the algebra generated by the horizontal endomorphisms J_Z. Recall the notion of H-type algebra from section 3.1.1.

Lemma 3.2.13. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation. Then for every* $p \in \mathbb{M}$ *the tangent space* $T_p\mathbb{M}$ *is an H-type algebra.*

Proof. The proof follows immediately from consideration of the H-type condition eq. (3.2.1) pointwise. $\square$

By a polarization argument, we see that for all $Z_1, Z_2 \in \mathcal{V}$ we have the Clifford relation

$$J_{Z_1} J_{Z_2} + J_{Z_2} J_{Z_1} = -g(Z_1, Z_2)\mathrm{Id}.$$

By the universal property of Clifford algebras we can extend the map $J \colon \Gamma(\mathcal{V}) \to \mathrm{End}(\Gamma(\mathcal{H}))$ defined by $Z \mapsto J_Z$ to a representation of $\mathbf{Cl}(\mathcal{V}_p)$ for $p \in \mathbb{M}$. In particular, at any $p \in \mathbb{M}$ such a map can be uniquely extended into a bundle algebra homomorphism from the Clifford algebra $\mathbf{Cl}(\mathcal{V}_p)$ to the algebra of horizontal endomorphisms $\mathrm{End}(\mathcal{H}_p)$, where the product on $\mathrm{End}(\mathcal{H}_p)$ is given by composition. That is

$$J_1 = \mathbf{Id}_{\mathcal{H}} \text{ and } J_{v \cdot w} = J_v J_w.$$

We make the identification $\Lambda^2 \mathcal{V} \cong \mathbf{Cl}_2(\mathcal{V}) \subset \mathbf{Cl}(\mathcal{V})$ obtained through the canonical isomorphism $Z_1 \wedge Z_2 \mapsto Z_1 \cdot Z_2 + \langle Z_1, Z_2 \rangle$.

In the remainder of this section, we consider various properties of this homomorphism.

Quaternionic Structures

We first consider the algebra generated by the J maps. That is,

Definition 3.2.14. Let $(\mathbb{M}, \mathcal{H}, g)$ be a H-type foliation. For $p \in \mathbb{M}$, denote by $\mathfrak{a}(p)$ the Lie sub-algebra of $\mathrm{End}(\mathcal{H}_p)$ generated by the $J_z, z \in \mathcal{V}_p$.

It turns out that there are relatively few possibilities, and these are strongly determined by $\mathcal{V}$.

Lemma 3.2.15 ([24, Lemma 2.12])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation. Let* $p \in \mathbb{M}$*. Consider* $\operatorname{End}(\mathcal{H}_p)$ *as a Lie algebra with commutator brackets. One of the following holds:*

(i) $m = 1$ *and* $\mathfrak{a}(p) = \{J_z \colon z \in \mathcal{V}_p\} \cong \mathbb{R}$*;*

(ii) $m = 3$ *and* $\mathfrak{a}(p) = \{J_z \colon z \in \mathcal{V}_p\} \cong \mathfrak{so}(3)$*;*

(iii) $m \geq 2$ *and* $\mathfrak{a}(p) = \{J_{z_0}, [J_{z_1}, J_{z_2}] \colon z_1, z_2 \in \mathcal{V}_p\} \cong \mathfrak{so}(m+1)$*.*

Proof. We refer to [24] for the proof, but observe that the key consideration is that for $m = 3$ it can hold that for orthogonal $z_1, z_2, z_3 \in \mathcal{V}_p$ that $J_{12} = J_3$; this is case (ii). If $m \geq 2$ and (ii) doesn't hold then the J_z do not form a Lie algebra without the brackets $[J_{z_1}, J_{z_2}]$.

□

Case (ii) is often singular in proofs to follow in a manner that is somehow analogous to the case of self-dual Einstein manifolds; we therefore distinguish it.

Definition 3.2.16. Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that $(\mathbb{M}, \mathcal{H}, g)$ is of quaternionic type if case (ii) of lemma 3.2.15 holds.

Remark 3.2.17. For H-type foliations of quaternionic type it must be that the field $\mathfrak{A}(p) = \{J_z \colon z \in \mathbb{R} \oplus \mathcal{V}\}$ is field isomorphic to the quaternions, which motivates the name.

While the definition of $\mathfrak{a}(p)$ is only sensible pointwise, it turns out to be independent of the choice of point.

Lemma 3.2.18 ([24, Lemma 2.13])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with horizontally parallel torsion. Then for any* $p, q \in \mathbb{M}$*,* $\mathfrak{a}(p)$ *is isomorphic to* $\mathfrak{a}(q)$*.*

Proof. Since $\nabla_{\mathcal{H}} J = 0$ follows from the horizontally parallel torsion, the ∇-parallel transport along any horizontal curve connecting p to q induces a Lie algebra isomorphism $\mathfrak{a}(p) \cong \mathfrak{a}(q)$. □

The J^2 condition

Instead of examining the Lie algebra $\mathfrak{a}$, we can consider instead a condition on the composition of J maps.

Definition 3.2.19. Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that the J^2 condition holds if for any orthogonal $Z_1, Z_2 \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$ it holds that there exists a $Z_3 \in \Gamma(\mathcal{V})$ such that

$$J_{Z_1} J_{Z_2} X = J_{Z_3} X$$

It should be emphasized, Z_3 can depend on Z_1, Z_2, and X, and so this does not imply that $\mathfrak{a}(p)$ form a subalgebra of $\mathrm{End}(\mathcal{H}_p)$. However,

Lemma 3.2.20. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion and satisfying the J^2 condition. Then one of the following occurs:*

- $m = 1$, $\mathcal{H}_p \cong \mathbb{C}^{n/2}$
- $m = 3$, $\mathcal{H}_p \cong \mathbb{H}^{n/4}$
- $m = 7$, $\mathcal{H}_p \cong \mathbb{O}^1$

for each point $p \in \mathbb{M}$.

Proof. It is known from [52, 45] that the J^2 condition for H-type algebras implies the theorem for each p. The lemma follows from lemma 3.2.18. □

Parallel Horizontal Clifford Structures

We can also consider how the Bott connection ∇ can interact with the J maps. In particular, we have the following definition.

Definition 3.2.21. Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. We say that $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation with parallel horizontal Clifford structure if there exists a smooth bundle map $\Psi : \mathcal{V} \times \mathcal{V} \to \mathbf{Cl}_2(\mathcal{V})$ such that for every $Z_1, Z_2 \in \Gamma(\mathcal{V})$

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}.$$

Essentially, the existence of a parallel horizontal Clifford structure is equivalent to the existence of a subgroup of $\mathrm{End}(\mathcal{H})$ (isomorphic to $\mathbf{Cl}(\mathcal{V})$) preserved by horizontal parallel transport. We defer consideration of this idea to section 3.4.

3.2.3 Curvature Dimension Inequalities

Curvature Dimension Inequalities

In Riemannian geometry there are many results that depend on a lower bound on the Ricci curvature; that is, we say that $\rho \in \mathbb{R}$ is a lower Ricci curvature bound on a Riemannian manifold $(\mathbb{M}, g)$ if it holds for any $X \in T\mathbb{M}$ that

$$\mathrm{Ric}(X, X) \geq \rho g(X, X).$$

These go back as far as [37]; standard references such as [59, 96, 77, 80, 83, 116] give plethora examples.

It was shown in [9] that a lower Ricci bound $\mathrm{Ric} \geq \rho g$ is equivalent to the statement

$$\|\nabla^2 f\|^2 + \mathrm{Ric}(\nabla f, \nabla f) \geq \frac{1}{n}(\Delta f)^2 + \rho\|\nabla f\|^2$$

for any $f \in C^\infty(\mathbb{M})$, where $\mathbb{M}$ has dimension n. Remarkably many classical Riemannian results relying on Ricci lower bounds can be derived directly from this, using purely analytical considerations.

Bakry, Ledoux, and their coauthors [10, 82] generalized this idea, allowing for a recovery of many of these classical Riemannian results on spaces that don't have a natural notion of curvature. In particular, they associate to a smooth, second-order diffusion operator L with real coefficients satisfying $L1 = 0$ the symmetric forms

$$\Gamma(f,g) = \frac{1}{2}\Big(L(fg) - fLg - gLf\Big)$$
$$\Gamma_2(f,g) = \frac{1}{2}\Big(L\Gamma(fg) - \Gamma(f,Lg) - \Gamma(g,Lf)\Big).$$

In the particular case $L = \Delta$, the Riemannian Laplacian, we see that

$$\Gamma(f) := \Gamma(f,f) = \|\nabla f\|^2, \qquad \Gamma_2(f) := \Gamma_2(f,f) = \|\nabla^2 f\|^2 + \mathrm{Ric}(\nabla f, \nabla f)$$

and we can rewrite the inequality as

$$\Gamma_2(f) \geq \frac{1}{n}(Lf)^2 + \rho\Gamma(f).$$

This is referred to as the <u>curvature dimension inequality</u> $CD(\rho, n)$. As in the special case of Riemannian manifolds, this condition allows for the recovery of a wide array of results from purely analytical considerations.

On an H-type foliation $(\mathbb{M}, \mathcal{H}, g)$ we define the sub-Laplacian Δ_{sR} as the generator of the symmetric closable bilinear form in $L^2(\mathbb{M}, \mu_g)$:

$$\mathcal{E}_{\mathcal{H}}(u,v) = \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} u, \nabla_{\mathcal{H}} v \rangle \, d\mu_g, \quad u,v \in C_0^\infty(\mathbb{M})$$

where the horizontal gradient $\nabla_{\mathcal{H}}$ denotes the projection of the Levi-Civita connection ∇^g onto $\mathcal{H}$. We define similarly the vertical gradient $\nabla_{\mathcal{V}}$. The bracket-generating condition on $\mathcal{H}$ implies Δ_{sR} is locally subelliptic [69], and completeness of the Riemannian metric g implies Δ_{sR} is essentially self-adjoint on $C_0^\infty(\mathbb{M})$ [17].

We also define the horizontal Laplacian $\Delta_{\mathcal{H}}$ as the horizontal trace of the Ricci tensor; that is

$$\Delta_{\mathcal{H}} f = \sum_{i=1}^{n} \mathrm{Hess}(f)(X_i, X_i)$$

for an orthonormal basis X_i of $\mathcal{H}$. On H-type foliations the Riemannian measure μ_g is proportional to the intrinsic Popp's measure, and so $\Delta_{\mathcal{H}}$ and Δ_{sR} coincide [13].

Unfortunately, setting $L = \Delta_{\mathcal{H}}$, it is too much to hope that a curvature dimension inequality will hold; this can be seen as a consequence of lemma 1.2.3. Addressing precisely this deficiency, Baudoin and Garofalo introduced in [21] the following generalization.

Say a symmetric bilinear form $\Gamma^Z\colon C^\infty(\mathbb{M}) \times C^\infty(\mathbb{M}) \to \mathbb{R}$ is admissible if it obeys the following conditions:

(i) There exists an increasing sequence $h_k \in C_0^\infty(\mathbb{M})$ such that $h_k \nearrow 1$ on $\mathbb{M}$ and

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \to 0 \text{ as } k \to \infty.$$

(ii) For any $f \in C^\infty(\mathbb{M})$ it holds that $\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f))$.

Denote

$$\Gamma_2(f,g) = \frac{1}{2}\Big(L\Gamma^Z(f,g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)\Big).$$

Definition 3.2.22. We say that $\mathbb{M}$ satisfies the generalized curvature dimension inequality $GCD(\rho_1, \rho_2, \kappa, n)$ with respect to L and admissible Γ^Z if there exist constants $\rho_1 \in \mathbb{R}, \rho_2 > 0, \kappa \geq 0$, and $0 < n \leq +\infty$ such that

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{n}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f)$$

holds for all $f \in C^\infty(\mathbb{M})$ and $\nu > 0$.

Notably, the bilinear map Γ^Z is not intrinsic to the space being considered. In fact, on a Riemannian manifold $(\mathbb{M}, g)$ setting $L = \Delta, \Gamma^Z = 0, \kappa = 0$ we recover the curvature dimension inequality $CD(\rho_1, n)$.

We will define on H-type foliations

$$\Gamma^Z(f,g) = g(\nabla_{\mathcal{V}} f, \nabla_{\mathcal{V}} g)$$

and so we see that Γ^Z is a measure of the contribution of $\mathcal{V}$; considering that we are interested in a sub-Riemannian result, it's sensible that this should not be expected to be intrinsic.

Importantly, it has been shown that on a sub-Riemannian manifold satisfying the generalized curvature dimension inequality that many important Riemannian results can be recovered analogously to the curvature dimension inequality; in particular the GCD is known to imply the following:

- Li-Yau type inequality
- Scale-invariant parabolic Harnack inequality
- Off-diagonal Gaussian upper bounds
- Liouville-type theorem
- Bonnet-Meyers diameter bound

Essentially, we can regard the $GCD(\rho_1, \rho_2, \kappa, d)$ as a candidate for replacing a lower Ricci bound. For full details, see [21, 18].

We will proceed to show that on H-type foliations a generalized curvature dimension inequality holds given a lower bound on the horizontal Ricci curvature.

Theorem 3.2.23 ([24, Proposition 2.20])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be a complete H-type foliation, and denote* $n = \text{rank}(\mathcal{H}), m = \text{rank}(\mathcal{V})$*. If there exists a constant* $\rho \in \mathbb{R}$ *such that* $\text{Ric}_{\mathcal{H}} \geq \rho g_{\mathcal{H}}$*, then the* $GCD(\rho, \frac{n}{4}, m, n)$ *is satisfied.*

From physics, there is a notion called the Yang-Mills property that we will need. We see that all H-type foliations naturally satisfy it.

Lemma 3.2.24 ([24, Theorem 2.17])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation. Then*

$$(\delta_{\mathcal{H}} T) = \text{Tr}_{\mathcal{H}}(\nabla^B_{\times} T)(\times, \cdot) = 0.$$

We say that $(\mathbb{M}, \mathcal{H}, g)$ *is Yang-Mills.*

Proof. Let $p \in \mathbb{M}$ be arbitrary. Let $Z \in \mathcal{V}_p$ be a unit vector and $X_1, \ldots, X_n$ be an orthonormal basis of $\mathcal{H}_p$. By lemma 3.2.7 so is $J_Z X_1, \dots, J_Z X_n$. For $Y \in \mathcal{H}_p$,

$$\begin{aligned} g(\mathrm{Tr}_{\mathcal{H}}(\nabla_\times T)(\times, Y), Z) &= \sum_{i=1}^n g((\nabla_{J_Z X_i} J)_Z J_Z X_i, Y) \\ &= -\sum_{i=1}^n g(\nabla_{X_i} J)_Z X_i, Y) \\ &= -g(\mathrm{Tr}_{\mathcal{H}}(\nabla_\times T)(\times, Y), Z). \end{aligned}$$

Where we use the fact that for $Z \in \Gamma(\mathcal{V})$,

$$(\nabla_{J_Z X} J)_Z J_Z X = -\|Z\|^2 (\nabla_X J)_Z X$$

which follows from a clever application of the Bianchi identity [24, lemma 2.18]. We see then that $\delta_{\mathcal{H}} T = \mathrm{Tr}_{\mathcal{H}}(\nabla_\times T)(\times, \cdot) = 0$ and so the foliation is Yang-Mills. □

Proof of theorem 3.2.23. We first define

$$\begin{aligned} \mathcal{R}(f) &= \mathrm{Ric}(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} f) - (\delta_{\mathcal{H}} T)(\nabla_{\mathcal{H}} f) f + \frac{1}{4} \sum_{\ell,k=1}^n (T(X_\ell, X_k) f)^2 \\ \mathcal{S}(f) &= -2 \sum_{i=1}^n g(\nabla_{X_i} \nabla_{\mathcal{V}} f, T(X_i, \nabla_{\mathcal{H}} f)) \\ \mathcal{T}(f) &= \sum_{i=1}^n \|T(X_i, \nabla_{\mathcal{H}} f)\|^2 \end{aligned}$$

where the X_i are any orthonormal basis of $\mathcal{H}$.

Lemma 3.2.25. *On a sub-Riemannian manifold* $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ *with* $n = \mathrm{rank}(\mathcal{H})$, *the system*

$$\begin{cases} \mathcal{R}(f) \geq \rho_1 \Gamma(f) + \rho_2 \Gamma^Z(f), \\ \mathcal{T}(f) \leq \kappa \Gamma(f) \end{cases}$$

implies that $GCD(\rho_1, \rho_2, \kappa, n)$ *holds for the sub-Laplacian* Δ_{sR}.

Proof. This is [21, theorem 2.19]; first one shows that there hold Bochner-type formulas

$$\Gamma_2(f) = \|\nabla^2_{\mathcal{H}} f\|^2 + \mathcal{R}(f) + \mathcal{S}(f)$$
$$\Gamma_2^Z(f) = \|\nabla_{\mathcal{H}} \nabla_{\mathcal{V}} f\|^2,$$

which can be computed explicitly in an adapted frame. The lemma then follows from several clever applications of Schwarz' inequality. □

We can expand $T(X_\ell, X_k) = \sum_{i=1}^m g_{\mathcal{H}}(J_{Z_i} X_\ell, X_k) Z_i$ for an orthonormal basis Z_i of $\mathcal{V}$, and so

$$\sum_{\ell,k=1}^{n} (T(X_\ell, X_k)f)^2 = \sum_{\ell,k=1}^{n} (g_{\mathcal{H}}(J_{\nabla_{\mathcal{V}} f} X_\ell, X_k))^2 = n\|\nabla_{\mathcal{V}} f\|^2. \tag{3.2.2}$$

By lemma 3.2.24 we see that the horizontal divergence of the torsion $\delta_{\mathcal{H}} T$ vanishes, and together with eq. (3.2.2) this implies

$$\mathcal{R}(f) \geq \rho\Gamma(f) + \frac{n}{4}\Gamma^Z(f).$$

Finally, we expand

$$\mathcal{T}(f) = \sum_{i=1}^{n}\sum_{j=1}^{m} \left\| g_{\mathcal{H}}(J_{Z_j} \nabla_{\mathcal{H}} f, X_i)\right\|^2 = \sum_{j=1}^{m} \|J_{Z_j} \nabla_{\mathcal{H}} f\|^2 = m\Gamma(f)$$

for an orthonormal basis Z_j of $\mathcal{V}$. With lemma 3.2.25 this completes the proof. □

Remark 3.2.26. We see that for the curvature quantity

$$\mathcal{R}(f) = \mathrm{Ric}(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} f) - (\delta_{\mathcal{H}} T)(\nabla_{\mathcal{H}} f) f + \frac{1}{4} \sum_{\ell,k=1}^{n} (T(X_\ell, X_k) f)^2$$

the only term with both vertical and horizontal derivatives is the horizontal divergence $(\delta_{\mathcal{H}} T)(\nabla_{\mathcal{H}} f) f$; the fact that this vanishes on Yang-Mills manifolds is the essential reason why we can separately bound the horizontal and vertical derivatives in lemma 3.2.25.

Remark 3.2.27. As shown in [21, theorem 2.20], the system of lemma 3.2.25 is actually equivalent to $GCD(\rho_1, \rho_2, \kappa, n)$, but this takes significantly more work to show.

We list some immediate results for H-type foliations that follow from established consequences of the generalized curvature dimension inequality.

Corollary 3.2.28 ([24, Corollary 2.21])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be a complete H-type foliation with* $\mathrm{Ric}_{\mathcal{H}} \geq \rho g_{\mathcal{H}}$ *for some* $\rho \in \mathbb{R}$*. Let us denote by* d_{cc} *the Carnot-Carathéodory distance.*

1. *If* $\rho \geq 0$*, then the metric measure space* $(\mathbb{M}, d_{cc}, \mu)$ *satisfies the volume doubling property and supports a 2-Poincaré inequality, i.e. there exist constants* $C_D, C_P > 0$*, depending only on* ρ, n, m*, for which one has for every* $p \in \mathbb{M}$ *and every* $r > 0$*:*

$$\mu(B(p, 2r)) \leq C_D \ \mu(B(p, r)),$$

$$\int_{B(p,r)} |f - f_B|^2 d\mu_g \leq C_P r^2 \int_{B(p,r)} \|\nabla_{\mathcal{H}} f\|^2 d\mu_g,$$

for every $f \in C^1(B(p,r))$*, where we have let* $f_B = \mu_g(B)^{-1} \int_B f d\mu_g$*, with* $B = B(p,r)$*.*

2. *If* $\rho > 0$*, then* $\mathbb{M}$ *is compact with a finite fundamental group and*

$$\mathbf{diam}(\mathbb{M}, d_{cc}) \leq 2\sqrt{3}\pi\sqrt{\frac{(n+4m)(n+6m)}{n\rho}}.$$

3. *If* $\rho > 0$*, then the first non zero eigenvalue of the sub-Laplacian* $-\Delta_{\mathcal{H}}$ *satisfies*

$$\lambda_1 \geq \frac{n\rho}{n+3m-1}.$$

Proof. Point 1 follows from [18, Theorem 1.5], and Point 2 from [21, Theorem 10.1] or [17, Theorem 6.1] for a simpler proof. Point 3 follows from [17, Theorem 4.9]. □

3.3 Some specific H-type Foliations

In this section we make explicit some important examples of H-type foliations, both to demonstrate the considerable number of sub-Riemannian spaces included by this definition as well as to provide the reader with a reference point by which to think of these structures.

3.3.1 Heisenberg Groups, Hopf fibrations, and Anti-de Sitter Spaces

The prototypical example of a sub-Riemannian manifold is the Heisenberg group, associated to flat Euclidean space.

Definition 3.3.1. Let $\mathbb{R}^{2n+1}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ be equipped with the foliation

$$\mathbb{R} \hookrightarrow \mathbb{R}^{2n+1} \xrightarrow{\pi} \mathbb{C}^n.$$

where $\mathcal{V}$ tangent to the fibers is generated by $Z = \partial_z$ and define a transversal horizontal distribution $\mathcal{H} = \text{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$, where

$$X_i = \partial_{x_i} - \frac{1}{2} y_i \partial_z \qquad Y_i = \partial_{y_i} + \frac{1}{2} x_i \partial_z.$$

Defining a Riemannian metric g so that the vectors X_i, Y_i, Z are orthonormal, we have that $(\mathbb{R}^{2n+1}, \mathcal{H}, g)$ is an H-type foliation called the Heisenberg group.

This object (for $n = 1$) is the original motivation for the notion of H(eisenberg)-type groups [76, 79]. It arises naturally in physics, as horizontal curves in this space describe the motion of electrons through an electric field. From the sub-Riemannian perspective, we see this as the model "flat space". There are many references, see e.g. [88, 45]. The thesis [90] explicitly describes foliations of the Heisenberg group.

The notion of curvature on H-type foliations (or sub-Riemannian geometry more generally) is subtle; we will explore throughout the rest of the thesis. There are two other particular H-type foliations that we want to keep in mind as models of positive and negative curvature.

To model positive curvature we have the Hopf fibration, associated to the sphere S^{2n+1}.

Definition 3.3.2. Identify S^{2n+1} with the set of points $z \in \mathbb{C}^{n+1}$ with $\|z\| = 1$.

There is a natural S^1 action on S^{2n+1} given by

$$(z_1, \dots, z_{n+1}) \mapsto (e^{i\theta} z_1, \dots, e^{i\theta} z_{n+1})$$

which induces the submersion

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n.$$

we have that $(S^{2n+1}, \mathcal{H}, g)$ is an H-type foliation called the complex Hopf fibration or sometimes the CR sphere. See [55, 95] for more details; in [27] the heat kernel for the sub-Laplacian is explicitly computed.

Analogously we have the model of negative curvature, the Anti-de Sitter space, associated to the hyperbolic space $\mathbf{H}^{2n+1}$.

Definition 3.3.3. Identify $\mathbf{H}^{2n+1}$ with the set of points $z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1}$ with $\|z\|_{\mathbf{H}} := \sum_{j=1}^{n} \|z_j\|^2 - \|z_{n+1}\|^2 = -1$. We have the natural S^1 action

$$(z_1, \dots, z_{n+1}) \mapsto (e^{i\theta} z_1, \dots, e^{i\theta} z_{n+1})$$

which induces the submersion

$$S^1 \hookrightarrow \mathbf{H}^{2n+1} \xrightarrow{\pi} \mathbb{C}H^n.$$

we have that $(\mathbf{H}^{2n+1}, \mathcal{H}, g)$ is an H-type foliation called the Anti-de Sitter (AdS) fibration. For more details see [43, 49, 95]; in [112] a heat kernel for the sub-Laplacian is explicitly computed.

We can analogously extend these constructions to the quaternions and octonions as in table 3.3.1.

Manifold	Fibration	References
Quaternionic Hopf fibration	$S^3 \hookrightarrow S^{4n+3} \to \mathbb{H}P^n$	[29]
Quaternionic Anti-de Sitter fibration	$S^3 \hookrightarrow \mathbf{H}^{4n+3} \to \mathbb{H}H^n$	[43] [20]
Octonionic Hopf fibration	$S^7 \hookrightarrow S^{15} \to \mathbb{O}P^1$	[94] [19]
Octonionic Anti-de Sitter fibration	$S^7 \hookrightarrow \mathbf{H}^{15} \to \mathbb{O}H^1$	[43]

Table 3.3.1: Model Quaternionic and Octonionic fibrations

Notice, because of the rigidity of the octonions, their associated fibrations can only exist over $\mathbb{O}P^1$ and $\mathbb{O}H^1$.

3.3.2 Contact and 3K-Contact Manifolds

Recall definition 2.2.6; a $2n+1$-dimensional manifold $\mathbb{M}$ equipped with a differential form η such that $\eta \wedge (d\eta)^n$ is a volume form is called a contact manifold.

Proposition 3.3.4. *Let* $(\mathbb{M}, \eta, g)$ *be a contact manifold with compatible Riemannian metric. Define* $\mathcal{H} = \ker \eta$*. Then* $(\mathbb{M}, \mathcal{H}, g)$ *is an H-type foliation up to a choice of normalization, see remark 3.2.3.*

This follows from the fact that the nonvanishing condition on $\eta \wedge (d\eta)^n$ is precisely the necessary condition for $\mathcal{H}$ to be bracket-generating, see [69]. The first examples of the last section fall under this category. Specifically, we have the contact structures in table 3.3.2.

We can extend the idea of these constructions to the case $\text{rank}(\mathcal{V}) = 3$ by considering $4n+3$-dimensional manifolds equipped with an $\mathbb{R}^3$-valued differential form $\eta = (\eta_1, \eta_2, \eta_3)$ constructed from a triple of contact forms. This is analogous to the

Manifold	Contact Structure
Heisenberg group $\mathbb{R}^{2n+1}$	$\eta = dz - \frac{1}{2}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$
Complex Hopf fibration S^{2n+1}	$\eta = \frac{i}{2}\sum_{j=1}^{n+1}\left(\overline{z_j}dz_j - z_j d\overline{z_j}\right)$
Anti-de Sitter fibration $\mathbf{H}^{2n+1}$	$\eta = \frac{i}{2}\left(\sum_{j=1}^{n}\left(\overline{z_j}dz_j - z_j d\overline{z_j}\right) - \left(\overline{z_{n+1}}dz_{n+1} - z_{n+1}d\overline{z_{n+1}}\right)\right)$

Table 3.3.2: Contact structures on model fibrations

situation of definition 2.2.26. Under suitable compatibility conditions we recover 3K-contact manifolds [72, 108], especially the Quaternionic Heisenberg group, and the Hopf and Anti-de Sitter fibrations as in table 1.3.1.

3.3.3 Twistor Spaces

Because contact structures naturally generate a representation of the complex numbers, quaternions, and octonions, these will only give examples in co-dimension 1, 3, or 8. However, H-type foliations do allow for $m = \mathrm{rank}(\mathcal{V})$ to take any dimension. In the case $m = 2$, we have the fascinating example of twistor spaces.

Definition 3.3.5. Let $(\mathbb{B}, j)$ be a quaternionic-Kähler manifold [35, Chapter 14] of dimension $4k \geq 8$. Consider the subbundle $E \subseteq \mathrm{End}(T\mathbb{M})$ spanned by a triple of complex structures I, J, K. Define an inner product g_E by setting these structures to be orthonormal, and fix $\rho > 0$. The subbundle $\mathcal{Z} \subset E$ determined pointwise as the sphere of constant g_E-radius ρ forms the twistor bundle over $\mathbb{B}$

$$S^2 \hookrightarrow \mathcal{Z} \to \mathbb{B}.$$

This is explored in [65, 105], and analogous notions arise over projective and hyperbolic spaces. These are H-type foliations, and appear on table 1.3.1.

3.4 Parallel Horizontal Clifford Structures

In this section, we consider the covariant derivatives of J; that is, we investigate what can we understand of the structure of an H-type foliation from the properties of

$$(\nabla_X J)_Y = (\nabla_X T)^\flat(\cdot, \cdot^\sharp, Y).$$

This is of particular interest because, as we will come to see, there is a relationship between the Riemann curvature tensor and the covariant derivatives of J_Z. As discussed in section 3.2.2 we can extend J to a representation $J\colon \mathbf{Cl}(\mathcal{V}) \to \mathrm{End}(\mathcal{H})$, and thus the algebraic structure of $\mathbf{Cl}(\mathcal{V})$ can influence the curvature properties of $\mathcal{H}$.

We see that $(\nabla_X J)_Y$ vanishes in the case of completely parallel torsion; in the case or horizontally parallel torsion the quantity is nontrivial only if $X, Y \in \mathcal{V}$. We recall the identification $\wedge^2\mathcal{V}$ with $\mathbf{Cl}_2(\mathcal{V})$ from section 3.2.2 and definition 3.2.21.

Definition 3.4.1. Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. We say that $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation with parallel horizontal Clifford structure if there exists a smooth bundle map $\Psi : \mathcal{V} \times \mathcal{V} \to \mathbf{Cl}_2(\mathcal{V})$ such that for every $Z_1, Z_2 \in \Gamma(\mathcal{V})$

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}.$$

Remark 3.4.2. If $m = 1$, then the parallel horizontal Clifford assumption is always satisfied with $\Psi = 0$.

This definition is motivated by the analogous notion of Clifford structure on Riemannian manifolds; these were completely characterized by Moroianu and Semmelmann in [89]. We summarize the algebraic consequences for Ψ in the following theorem.

Theorem 3.4.3 ([24, Theorem 3.6])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with parallel horizontal Clifford structure. Then there exists a constant* $\kappa \in \mathbb{R}$ *such that* Ψ *has the form*

$$\Psi(u, v) = -\kappa(u \cdot v + g(u, v)).$$

and the sectional curvature of the leaves of the foliation associated to $\mathcal{V}$ *is constant equal to* κ^2*. If the torsion is completely parallel, the leaves are flat.*

From this we see that parallel horizontal Clifford structures are fairly rigid, and thereby give significant information relating $\mathbf{Cl}(\mathcal{V})$ to the $\mathrm{End}(\mathcal{H})$ generated by the J_Z.

Proof. We refer to [24, theorem 3.6] for the complete proof, but remark that the the essential steps are in recognizing that the symmetries

1. $\Psi(u, v) = -\Psi(v, u)$;

2. $\Psi(u, v) \cdot v + v \cdot \Psi(u, v) = 0$.

hold for all $u, v \in \mathcal{V}_p$. The form of Ψ follows from consideration of the possible homomorphisms $\Lambda^2\mathcal{V} \to \mathbf{Cl}_2(\mathcal{V})$ and showing that any other terms vanish due to the symmetries. That κ^2 gives the vertical sectional curvature follows from

$$\mathrm{Sec}(u \wedge v) = g(R(u, v)v, u) = \|(\nabla_u J)_v w\|^2 = \kappa^2$$

for any orthonormal $v, u \in \mathcal{V}_p$ and unit $w \in \mathcal{H}_p$, where we applied the first result. $\square$

3.4.1 H-type foliations with completely parallel torsion

We begin with the simplest case, that of H-type foliations with completely parallel torsion. In this setting $(\nabla_X J)_Y$ vanishes, and so we always have parallel horizontal Clifford structure $\Psi = 0$.

Theorem 3.4.4 ([24, Theorem 3.8]). *Let $(\mathbb{M}, \mathcal{H}, g, \pi)$ be an H-type submersion with completely parallel torsion, and let the base space $(\mathbb{B}, j)$ be simply-connected. Then one of the following (non exclusive) cases occur:*

- $m = 1$, $\mathbb{M}$ *is Sasakian, and* $\mathbb{B}$ *is Kähler;*
- $m = 2$ *or* $m = 3$ *and* $\mathbb{B}$ *is locally hyper-Kähler;*
- m *is arbitrary,* $\mathbb{M}$ *is an H-type group, and* $\mathbb{B}$ *is flat and isometric to a representation of the Clifford algebra* $\mathbf{Cl}(\mathbb{R}^m)$.

Proof. We refer to [24] for the complete proof, but highlight here the essential idea that since the sectional curvature of $\mathcal{V}$ vanishes and we have a global submersion we can see that the maps $J_Z \in \mathrm{End}(\mathcal{H})$ project onto parallel almost complex structures $\overline{J}_Z$ on $\mathbb{B}$. Then for $m = 1$ we see that $\mathbb{B}$ is Kähler and for $m \geq 2$ we see that $\mathbb{B}$ is locally hyper-Kähler. When $m \geq 4$ considerations of holonomy force $\mathbb{B}$ to be flat and the theorem follows as a consequence of theorem 4.1.6 and theorem 4.1.7. $\square$

From this we see that the completely parallel torsion condition is rigid, and only allows for the H-type groups in dimension $m \geq 4$. The following corollary holds by observing that H-type foliations are always locally H-type submersions.

Corollary 3.4.5 ([24, Corollary 3.9])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with completely parallel torsion. If* $m \geq 2$*, then* $\mathbb{M}$ *is horizontally Ricci flat, i.e.* $\mathrm{Ric}_{\mathcal{H}} = 0$*. If* $m \geq 4$*, then* $\mathbb{M}$ *is horizontally flat, i.e.* $R_{\mathcal{H}} = 0$*.*

3.4.2 Horizontal Einstein property

Definition 3.4.6. Let $(\mathbb{M}, \mathcal{H}, g)$ be a totally geodesic foliation. We say that $(\mathbb{M}, \mathcal{H}, g)$ is horizontally Einstein if there exists some constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}_{\mathcal{H}}(X, Y) = \lambda g_{\mathcal{H}}(X, Y),$$

for all $X, Y \in \Gamma(\mathcal{H})$, where $\mathrm{Ric}_{\mathcal{H}}$ is the horizontal Ricci tensor of the Bott connection.

In this section, we prove the following theorem:

Theorem 3.4.7 ([24, Theorem 3.16])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with a parallel horizontal Clifford structure and* $m \geq 2$*. Then*

- *if* $m \neq 3$ *or if* $m = 3$ *and* $(\mathbb{M}, \mathcal{H}, g)$ *is quaternionic,*

$$\mathrm{Ric}_{\mathcal{H}} = \kappa \left(\frac{n}{4} + 2(m-1) \right) g_{\mathcal{H}},$$

- *otherwise (when* $m = 3$ *and* $(\mathbb{M}, \mathcal{H}, g)$ *is not of quaternionic type) then at any point,* $\mathcal{H}$ *orthogonally splits as a direct sum* $\mathcal{H}^+ \oplus \mathcal{H}^-$ *and for* $X, Y \in \Gamma(\mathcal{H})$*,*

$$\mathrm{Ric}_{\mathcal{H}}(X, Y) = \kappa \left(\frac{n}{4} + 4 \right) \langle X, Y \rangle + \frac{\kappa}{4} (\dim \mathcal{H}^+ - \dim \mathcal{H}^-) \langle \sigma(X), Y \rangle,$$

where $\sigma = \mathbf{Id}_{\mathcal{H}^+} \oplus (-\mathbf{Id}_{\mathcal{H}^-})$*.*

Remark 3.4.8. In the special case $m = 3$, but $\mathbb{M}$ is not quaternionic, the splitting $\mathcal{H}_p = \mathcal{H}_p^+ \oplus \mathcal{H}_p^-$ is related to the case of self-dual manifolds in dimension 4. Notably, $\nabla_{\mathcal{H}}\sigma = 0$ and so the splitting is independent of the point p. In all other cases, $(\mathbb{M}, \mathcal{H}, g)$ is horizontally Einstein.

In the case $m = 2$, the fact that $(\mathbb{M}, \mathcal{H}, g)$ is horizontally Einstein is related to the fact that quaternion Kähler manifolds are Einstein manifolds (see Berger [33], Ishihara [71] or Besse [35, theorem 14.39]), and the algebraic structure of our proof below somehow parallels the one of Ishihara and Besse (in the choice of a special horizontal basis). The key lemma is the following:

Lemma 3.4.9 ([24, Lemma 3.18])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be a totally geodesic foliation with horizontally parallel torsion. For any* $X, Y \in \Gamma(\mathcal{H})$ *and* $Z \in \Gamma(\mathcal{V})$*, we have*

$$\big[R_{\mathcal{H}}(X,Y), J_Z\big] = (\nabla_{T(X,Y)}J)_Z + J_{(\nabla_Z T)(X,Y)}.$$

Proof. Write the Hessian operator for ∇ as $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$. Using that J is parallel in horizontal directions and that $R(X,Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X} + \nabla_{T(X,Y)}$, we observe that for $X, Y \in \Gamma(\mathcal{H})$ we have

$$R(X,Y)J = \nabla_{T(X,Y)}J.$$

However, for $W \in \Gamma(\mathcal{H})$ and $Z \in \Gamma(\mathcal{V})$, we can also write

$$\begin{aligned}(R(X,Y)J)_Z W &= R(X,Y)J_Z W - J_{R(X,Y)Z}W - J_Z R(X,Y)W \\ &= R_{\mathcal{H}}(X,Y)J_Z W - J_{(\nabla_Z T)(X,Y)}W - J_Z R_{\mathcal{H}}(X,Y)W.\end{aligned}$$

The result follows. □

This gives us an important structural result in our setting.

Lemma 3.4.10. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with a parallel horizontal Clifford structure and* $m \geq 2$. *Let* $Z_1, \ldots, Z_m$ *be a local vertical orthonormal frame. It will hold that*

$$\big[R_{\mathcal{H}}(X,Y), J_i\big] = \kappa \sum_{j=1, j\neq i}^{m} \Big(\langle J_j X, Y\rangle J_{ij} - \langle J_{ij} X, Y\rangle J_j \Big). \tag{3.4.1}$$

Proof. We first observe that from Lemma 3.4.9 together with the parallel horizontal Clifford structure assumption, one obtains that for every $X, Y \in \Gamma(\mathcal{H})$,

$$\begin{aligned}\big[R_{\mathcal{H}}(X,Y), J_i\big] &= (\nabla_{T(X,Y)} J)_{Z_i} + J_{(\nabla_{Z_i} T)(X,Y)} \\ &= -\kappa J_{T(X,Y)\cdot Z_i + \langle T(X,Y), Z_i\rangle} + J_{(\nabla_{Z_i} T)(X,Y)}.\end{aligned}$$

Then, we note that

$$T(X,Y)\cdot Z_i + \langle T(X,Y), Z_i\rangle = -\sum_{j=1, j\neq i}^{m} \langle J_j X, Y\rangle Z_i \cdot Z_j,$$

and that

$$J_{(\nabla_{Z_i} T)(X,Y)} = \sum_{j=1}^{m} J_{\langle (\nabla_{Z_i} T)(X,Y), Z_j\rangle Z_j} = \sum_{j=1}^{m} J_{\langle (\nabla_{Z_i} J)_{Z_j} X, Y\rangle Z_j} = -\kappa \sum_{j=1, j\neq i}^{m} \langle J_{ij} X, Y\rangle J_j.$$

Combining the previous expressions completes the lemma. □

We will also need the following lemma in the case $m = 3$ and $(\mathbb{M}, \mathcal{H}, g)$ is non-quaternionic.

Lemma 3.4.11 ([24, Lemma 3.19]). *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be a totally geodesic foliation with horizontally parallel torsion and* $m = 3$. *Let* Z_1, Z_2, Z_3 *be a local orthonormal frame of* $\mathcal{V}$. *Then* $(\mathbb{M}, \mathcal{H}, g)$ *is of quaternionic type if and only if* $J_{Z_1}J_{Z_2}J_{Z_3} \in \{-\mathbf{Id}_{\mathcal{H}}, \mathbf{Id}_{\mathcal{H}}\}$. *If* $(\mathbb{M}, \mathcal{H}, g)$ *is not of quaternionic type, then* $\sigma = J_{Z_1}J_{Z_2}J_{Z_3}$ *is a non-trivial horizontal isometry such that* $\sigma^2 = \mathbf{Id}_{\mathcal{H}}$ *and that commutes with* $J_{Z_1}, J_{Z_2}, J_{Z_3}$.

We reproduce the complete proof of theorem 3.4.7 from [24].

Proof of theorem 3.4.7. We fix i, and $j \neq i$. Note that J_i, J_j, J_{ij} satisfy the quaternion relations, $J_i^2 = J_j^2 = J_{ij}^2 = J_iJ_jJ_{ij} = -\mathbf{Id}_{\mathcal{H}}$. Choose a local orthonormal basis X_ℓ of $\mathcal{H}$ such that if X_ℓ is in the basis, so are $J_iX_\ell, J_jX_\ell, J_{ij}X_\ell$ (up to a $\pm$ sign); this can be done by lemma 3.2.7. We then compute for $X, Y \in \Gamma(\mathcal{H})$,

$$\begin{aligned}
\mathrm{Ric}_{\mathcal{H}}(X, J_iY) &= -\sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, X_\ell)J_iY, X_\ell\rangle \\
&= -\sum_{\ell=1}^{n} \langle [R_{\mathcal{H}}(X, X_\ell), J_i]Y, X_\ell\rangle - \sum_{\ell=1}^{n} \langle J_iR_{\mathcal{H}}(X, X_\ell)Y, X_\ell\rangle \\
&= -\sum_{\ell=1}^{n} \langle [R_{\mathcal{H}}(X, X_\ell), J_i]Y, X_\ell\rangle + \sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, X_\ell)Y, J_iX_\ell\rangle .
\end{aligned}$$

On one hand, one obtains from (3.4.1):

$$\begin{aligned}
\sum_{\ell=1}^{n} \langle [R_{\mathcal{H}}(X, X_\ell), J_i]Y, X_\ell\rangle &= \kappa \sum_{\ell=1}^{n} \sum_{j=1, j\neq i}^{m} \Big(\langle J_jX, X_\ell\rangle\langle J_{ij}Y, X_\ell\rangle - \langle J_{ij}X, X_\ell\rangle\langle J_jY, X_\ell\rangle\Big) \\
&= \kappa \sum_{j=1, j\neq i}^{m} \Big(\langle J_jX, J_{ij}Y\rangle - \langle J_{ij}X, J_jY\rangle\Big) \\
&= 2\kappa(m-1)\langle J_iX, Y\rangle.
\end{aligned}$$

On the other hand, noticing that the set of $-J_iX_\ell \otimes X_\ell$ and the set of $X_\ell \otimes J_iX_\ell$ will be identical as X_ℓ varies across the whole basis, one obtains

$$\sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, X_\ell)Y, J_i X_\ell\rangle = \frac{1}{2}\sum_{\ell=1}^{n} \Big(\langle R_{\mathcal{H}}(X, X_\ell)Y, J_i X_\ell\rangle - \langle R_{\mathcal{H}}(X, J_i X_\ell)Y, X_\ell\rangle \Big)$$
$$= \frac{1}{2}\sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, Y)X_\ell, J_i X_\ell\rangle ,$$

where the second equality follows from Bianchi's identity and symmetries of the curvature tensor. It therefore remains to compute $\sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, Y)X_\ell, J_i X_\ell\rangle$. We use the fact that the set of $X_\ell \otimes J_i X_\ell$ and the set of $J_j X_\ell \otimes J_{ij} X_\ell$ will be identical as X_ℓ varies across the whole basis to obtain

$$2\sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, Y)X_\ell, J_i X_\ell\rangle = \sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, Y)X_\ell, J_i X_\ell\rangle + \langle R_{\mathcal{H}}(X, Y)J_j X_\ell, J_{ij} X_\ell\rangle$$
$$= \sum_{\ell=1}^{n} \langle R_{\mathcal{H}}(X, Y)X_\ell, J_j J_{ij} X_\ell\rangle + \langle R_{\mathcal{H}}(X, Y)J_j X_\ell, J_{ij} X_\ell\rangle$$
$$= \sum_{\ell=1}^{n} - \langle J_j R_{\mathcal{H}}(X, Y)X_\ell, J_{ij} X_\ell\rangle + \langle R_{\mathcal{H}}(X, Y)J_j X_\ell, J_{ij} X_\ell\rangle$$
$$= \sum_{\ell=1}^{n} \langle [R_{\mathcal{H}}(X, Y), J_j]X_\ell, J_{ij} X_\ell\rangle .$$

Now, from (3.4.1):

$$\sum_{\ell=1}^{n} \langle [R_{\mathcal{H}}(X, Y), J_j]X_\ell, J_{ij} X_\ell\rangle$$
$$= \kappa \sum_{\ell=1}^{n} \sum_{k=1, k\neq j}^{m} \Big(\langle J_k X, Y\rangle \langle J_{jk} X_\ell, J_{ij} X_\ell\rangle - \langle J_{jk} X, Y\rangle \langle J_k X_\ell, J_{ij} X_\ell\rangle \Big).$$

If $k \neq i$, one has $\langle J_{jk} X_\ell, J_{ij} X_\ell\rangle = 0$ and if $k = i$, $\langle J_{jk} X_\ell, J_{ij} X_\ell\rangle = -1$. Therefore, one obtains:

$$\sum_{\ell=1}^{n} \langle [R_{\mathcal{H}}(X,Y), J_j] X_\ell, J_{ij} X_\ell \rangle$$
$$= -\kappa n \langle J_i X, Y \rangle - \kappa \sum_{k=1, k\neq j}^{m} \sum_{\ell=1}^{n} \langle J_{jk} X, Y \rangle \langle J_k X_\ell, J_{ij} X_\ell \rangle .$$

The analysis of the sum $\sum_{\ell=1}^{n} \langle J_k X_\ell, J_{ij} X_\ell \rangle$ will depend on m. If $m = 2$, then one has $\sum_{\ell=1}^{n} \langle J_k X_\ell, J_{ij} X_\ell \rangle = 0$, because one must have $k = i$. If $m \geq 4$, then one can pick an index s which is different from i, j and k so that by using invariance of the trace by a change a basis:

$$\sum_{\ell=1}^{n} \langle J_k X_\ell, J_{ij} X_\ell \rangle = \sum_{\ell=1}^{n} \langle J_k J_s X_\ell, J_{ij} J_s X_\ell \rangle = -\sum_{\ell=1}^{n} \langle J_k X_\ell, J_{ij} X_\ell \rangle .$$

Therefore $\sum_{\ell=1}^{n} \langle J_k X_\ell, J_{ij} X_\ell \rangle = 0$. Summarizing the above computations, one deduces that for $i \neq j \neq k$,

$$\mathrm{Ric}_{\mathcal{H}}(X, J_i Y) = \begin{cases} -2\kappa(m-1)\langle J_i X, Y\rangle - \frac{\kappa n}{4}\langle J_i X, Y\rangle, \quad if \quad m \neq 3 \\ -4\kappa\langle J_i X, Y\rangle - \frac{\kappa n}{4}\langle J_i X, Y\rangle + \frac{\kappa}{4}\langle J_{jk} X, Y\rangle \mathbf{Tr}_{\mathcal{H}}(J_i J_j J_k), \quad if \quad m = 3. \end{cases}$$

Therefore, substituting Y by $J_i Y$ one concludes

$$\mathrm{Ric}_{\mathcal{H}}(X, Y) = \begin{cases} 2\kappa(m-1)\langle X, Y\rangle + \frac{\kappa n}{4}\langle X, Y\rangle, \quad if \quad m \neq 3 \\ 4\kappa\langle X, Y\rangle + \frac{\kappa n}{4}\langle X, Y\rangle + \frac{\kappa}{4}\langle J_1 J_2 J_3 X, Y\rangle \mathbf{Tr}_{\mathcal{H}}(J_1 J_2 J_3), \quad if \quad m = 3. \end{cases}$$

By denoting $\sigma = J_1 J_2 J_3$, $\mathcal{H}^+$ the 1 eigenspace of σ and $\mathcal{H}^-$ the -1 eigenspace of σ, one can then concludes with Lemma 3.4.11. We note that $\sigma^2 = \mathbf{Id}_{\mathcal{H}}$, thus $\nabla_{\mathcal{H}} \sigma = 0$. □

3.4.3 Sub-Riemannian diameter and first eigenvalue estimates

Combining theorem 3.4.7 with the results of corollary 3.2.28, we obtain the following.

Theorem 3.4.12 ([24, Corollary 3.20]). *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with a parallel horizontal Clifford structure such that* $\kappa > 0$. *Then,* $\mathbb{M}$ *is compact with finite fundamental group. Moreover,*

- *If* $m \neq 3$ *or* $m = 3$ *and* $(\mathbb{M}, \mathcal{H}, g)$ *is of quaternionic type then we have the sub-Riemannian diameter bound*

$$\operatorname{diam}(\mathbb{M}, d_{cc}) \leq 4\sqrt{3}\frac{\pi}{\sqrt{\kappa}}\sqrt{\frac{(n+4m)(n+6m)}{n(n+8(m-1))}},$$

  and we have the following estimate for the first eigenvalue of the sub-Laplacian

$$\lambda_1 \geq \frac{\kappa}{4}\frac{n(n+8(m-1))}{n+3m-1}.$$

- *If* $m = 3$ *and* $(\mathbb{M}, \mathcal{H}, g)$ *is not of quaternionic type, then we have the sub-Riemannian diameter bound*

$$\operatorname{diam}(\mathbb{M}, d_{cc}) \leq 2\sqrt{6}\frac{\pi}{\sqrt{\kappa}}\sqrt{\frac{(n+12)(n+18)}{n(n+8)}},$$

  and we have the following estimate for the first eigenvalue of the sub-Laplacian

$$\lambda_1 \geq \frac{n\kappa}{2}.$$

Remark 3.4.13. Compare this with [97], in which it is shown that the estimate

$$\lambda_1 \geq \frac{n\pi^2}{\mathrm{diam}(\mathbb{M}, d_{cc})^2}$$

holds, which is sharp and agrees with our result on the complex and quaternionic Hopf fibrations.

Chapter 4

Holonomy of H-type Foliations

In this chapter we will explore the collection of endomorphisms of $\mathcal{H}_p$ induced by Bott-parallel transport around loops at a point p. This is an extension of the idea of holonomy from Riemannian geometry, where it is well-known that the structure of such groups has strong consequences for the geometry of the underlying manifold.

4.1 Riemannian holonomy

On a Riemannian manifold $(\mathbb{M}, g)$ equipped with a metric connection ∇, there is a notion of parallel transport section 2.1.1; given a curve $\gamma\colon [0,T] \to \mathbb{M}$ and a vector $x \in T_{\gamma(0)}\mathbb{M}$, we say that a vector field X such that $X(\gamma(0)) = x$ and $\nabla_{\dot{\gamma}} X = 0$ is a parallel transport for x along γ. This induces a family of isomorphisms

$$\tau_\gamma(t)\colon T_{\gamma(0)}\mathbb{M} \to T_{\gamma(t)}\mathbb{M}$$

for all $0 \leq t \leq T$. In particular, if γ is a loop at a fixed point $p \in \mathbb{M}$ this gives an endomorphism of $T_p\mathbb{M}$.

Definition 4.1.1. We denote by $\mathbf{Hol}(\mathbb{M}, p)$ the collection of endomorphisms of $T_p\mathbb{M}$ induced by ∇-parallel transport along loops at p. This forms a group by composition, and is called the holonomy group of $\mathbb{M}$ at p. Restricting to only the loops homotopic to the constant loop at p, we have the restricted holonomy group at p, $\mathbf{Hol}^0(\mathbb{M}, p)$.

Proposition 4.1.2. *Let $(\mathbb{M}, g)$ be a Riemannian manifold with metric connection ∇.*

- *If $\mathbb{M}$ is connected, $\mathbf{Hol}(\mathbb{M}, p)$ is independent of the choice of $p \in \mathbb{M}$, and so we write simply $\mathbf{Hol}(\mathbb{M})$.*
- *Every element of $\mathbf{Hol}(\mathbb{M})$ is an isometry.*

Proof. The proofs are straightforward, we refer to [77, chapter 2]. □

One can characterize the infinitesimal generators of the holonomy group in terms of the associated Riemann curvature tensor.

Theorem 4.1.3 (Ambrose-Singer [8])**.** *Let $(\mathbb{M}, g)$ be a Riemannian manifold with metric connection ∇. For $p \in \mathbb{M}$, the Lie algebra $\mathfrak{hol}(\mathbb{M}, p)$ of $\mathbf{Hol}(\mathbb{M}, p)$ is exactly the sub-algebra of $\mathfrak{so}(T_p\mathbb{M})$ generated by the elements $\tau_\gamma^{-1} \circ R(\tau_\gamma u, \tau_\gamma v) \circ \tau_\gamma$ where u, v run through $T_p\mathbb{M}$ and γ through $\mathcal{C}_p$.*

Proof. This is explained thoroughly by [35, theorem 10.58, note 10.59]. □

Remark 4.1.4. We emphasize a note made in [35], that this result remains true without modification for any metric connection ∇.

We can also simplify the study of holonomy groups to those that are irreducible.

Theorem 4.1.5 (deRham [53])**.** *Let* $(\mathbb{M}, g)$ *be a complete, simply connected Riemannian manifold. Then* $\mathbf{Hol}(\mathbb{M})$ *is reducible as a direct product if and only if* $(\mathbb{M}, g)$ *is reducible as a Riemannian product.*

Holonomy associated to Levi-Civita connection

In particular, the holonomy group $\mathbf{Hol}(\mathbb{M})$ associated to the Levi-Civita connection is referred to as the Riemannian holonomy group; there is a complete classification of the possible Riemannian holonomies, as follows.

Theorem 4.1.6 (Berger [33], Simons [106])**.** *Suppose* $(\mathbb{M}, g)$ *is a Riemannian manifold with irreducible* $\mathbf{Hol}^0(\mathbb{M})$*. Then one of the following (nonexclusive) cases occur:*

- $\mathbb{M}$ *is locally symmetric and is of rank at least 2, or*
- $\mathbf{Hol}^0(\mathbb{M})$ *acts transitively on the sphere.*

Simons' proof [106] relies on the Ambrose-Singer theorem 4.1.3. One assumes that $\mathbf{Hol}^0(\mathbb{M})$ does not act transitively on the sphere, and after a lengthy algebraic argument one concludes the proof using the equivalence of local symmetry with the statement $\nabla R = 0$ shown by Cartan. A modern, geometric proof based on the holonomy of submanifolds was established by Olmos [93].

The holonomy of symmetric spaces is rather involved, but is completely classified by a series of results due to Cartan [47, 48]. See [78] for an introduction to the holonomy of symmetric spaces.

In the nonsymmetric case, the list of possible holonomy groups is rather short. This is a consequence of a classification of Lie groups acting transitively on the sphere due to Cartan.

Theorem 4.1.7. *For an irreducible, simply connected, nonsymmetric Riemannian manifold manifold* $\mathbb{M}$*, one of the following cases occurs:*

$dim(\mathbb{M})$	$\mathbf{Hol}^0(\mathbb{M})$	*Type*
n *arbitrary*	$O(n)$	*Generic*
n *arbitrary*	$SO(n)$	*Oriented*
$n = 2m$	$U(m)$	*Kähler*
$n = 2m$	$SU(m)$	*Calabi-Yau*
$n = 4m$	$Sp(m) \cdot Sp(1)$	*Quaternion-Kähler*
$n = 4m$	$Sp(m)$	*Hyperkähler*
$n = 7$	G_2	G_2 *manifold*
$n = 8$	$Spin(7)$	$Spin(7)$ *manifold*

This is the complete list of Lie groups acting transitively on the sphere, with the exception of $T \cdot Sp(m)$ and $Spin(9)$. Topological considerations rule out $T \cdot Sp(m)$. $Spin(9)$ occurs, but only for symmetric spaces. The remainder have been shown to exist explicitly; in particular the existence of G_2 and $Spin(7)$ manifolds was difficult, but was established by Bryant in [41, 42]. See [35, chapter 10] for a complete discussion.

4.2 Adapted holonomy of foliations

In this section we introduce a notion of horizontal holonomy for foliations associated to adapted connections.

Let $(\mathbb{M}, \mathcal{H}, g)$ be a totally geodesic Riemannian foliation with bundle-like metric, and denote by ∇ an adapted connection. Denote the Bott connection by ∇. If $p \in \mathbb{M}$

and $\gamma\colon [0,T] \to \mathbb{M}$ is a piecewise-C^1 curve, let us denote by $\tau_\gamma\colon T_p\mathbb{M} \to T_{\gamma(p)}\mathbb{M}$ the ∇-parallel transport along γ. Since ∇ is metric, τ_γ is an element of the orthogonal group $\mathcal{O}(T_p\mathbb{M}, T_{\gamma(p)}\mathbb{M})$. Moreover, since ∇ preserves $\mathcal{H}$ and $\mathcal{V}$, one has

$$\tau_\gamma(\mathcal{H}_p) \subseteq \mathcal{H}_{\gamma(p)}, \quad \tau_\gamma(\mathcal{V}_p) \subseteq \mathcal{V}_{\gamma(p)}.$$

Therefore τ_γ induces an isometry $\tau_\gamma|_{\mathcal{H}} \in \mathcal{O}(\mathcal{H}_p, \mathcal{H}_{\gamma(p)})$ and an isometry $\tau_\gamma|_{\mathcal{V}} \in \mathcal{O}(\mathcal{V}_p, \mathcal{V}_{\gamma(p)})$.

Denote by $\mathcal{C}_p$ the set of piecewise-C^1 loops based at $p \in \mathbb{M}$. We introduce the following holonomy groups associated to the connection ∇.

Definition 4.2.1. Let $p \in \mathbb{M}$. We call the subgroups of $\mathcal{O}(\mathcal{H}_p)$ and $\mathcal{O}(\mathcal{V}_p)$ generated by the set of all $\tau_\gamma|_{\mathcal{H}}$ and $\tau_\gamma|_{\mathcal{V}}$, $\gamma \in \mathcal{C}_p$ the horizontal holonomy group at p denoted by $\mathbf{Hol}(\mathcal{H}, p)$ and the vertical holonomy group at p denoted by $\mathbf{Hol}(\mathcal{V}, p)$, respectively. When restricting to the subset $\mathcal{C}_p^0 \subseteq \mathcal{C}_p$ consisting only of loops homotopic to the identity, we get the restricted holonomy subgroups denoted by $\mathbf{Hol}^0(\mathcal{H}, p)$ and $\mathbf{Hol}^0(\mathcal{V}, p)$.

The horizontal holonomy groups are all isomorphic. This enables us to talk of the horizontal holonomy group of $(\mathbb{M}, \mathcal{H}, g)$ which we will denote $\mathbf{Hol}(\mathcal{H}), \mathbf{Hol}^0(\mathcal{H})$. Similarly, we can talk of $\mathbf{Hol}(\mathcal{V})$, and $\mathbf{Hol}^0(\mathcal{V})$.

Remark 4.2.2. These constructions are invariant by the rescaling g_ε.

We have the following theorem describing the infinitesimal generators of the holonomy groups.

Theorem 4.2.3. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. For $p \in \mathbb{M}$, the Lie algebra $\mathfrak{hol}(\mathcal{H}, p)$ of $\mathbf{Hol}(\mathcal{H}, p)$ is exactly the sub-algebra of*

$\mathfrak{so}(\mathcal{H}_p)$ generated by the elements $\tau_\gamma^{-1} \circ R_{\mathcal{H}}(\tau_\gamma u, \tau_\gamma v) \circ \tau_\gamma$ where u, v run through $\mathcal{H}_p$ and γ through $\mathcal{C}_p$.

Proof. This is a straightforward consequence of the Riemannian Ambrose-Singer theorem 4.1.3 and lemma 3.2.5. □

Remark 4.2.4. Analogously to remark 4.1.4, the theorem will remain true for any connection with horizontally parallel torsion. This is essential, as otherwise lemma 3.2.5 will not hold.

Corollary 4.2.5. *Suppose $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation with completely parallel torsion. If $m \geq 4$ then $R_{\mathcal{H}} = 0$ and so $\mathbf{Hol}(\mathcal{H}) = \mathrm{Id}$.*

4.3 Holonomy of H-type foliations

In this section we explore the horizontal holonomy of H-type foliations. In particular we relate the horiztonal holonomy of H-type submersions to the holonomy of the base space.

4.3.1 Holonomy of H-type submersions

Let $(\mathbb{M}, \mathcal{H}, g, \pi)$ be an H-type submersion with base space $(\mathbb{B}, j)$. Assume that $\mathbb{B}$ is connected. We write $\nabla^{\mathbb{M}}, \nabla^{\mathbb{B}}$ for the Levi-Civita connections on $\mathbb{M}$ and $\mathbb{B}$, and ∇ for the Bott connection on $(\mathbb{M}, \mathcal{H}, g)$.

In this section, we will characterize the holonomy of H-type submersions in terms of the holonomy of the associated base space. To begin, we will leverage the fact that we are working on global submersions by defining the notion of basic vector field that will be key to our study of the holonomy.

Definition 4.3.1. We say $X \in \Gamma(T\mathbb{M})$ is projectable if there exists $\bar{X} \in \Gamma(T\mathbb{B})$ such that $d\pi(X) = \bar{X}$. We will say X and $\bar{X}$ are π-related. Moreover, if $X \in \Gamma(\mathcal{H})$ is projectable we say X is basic.

Basic vector fields are ubiquitous in the study of submersions and foliations. See for example [35, 110, 87, 66].

Lemma 4.3.2. *Associated to any vector field $\bar{X} \in \Gamma(T\mathbb{B})$ there is an unique basic π-related vector field $X \in \Gamma(\mathcal{H})$.*

Lemma 4.3.3. *Let $X, Y \in \Gamma(T\mathbb{M})$ be projectable, and let $Z \in \Gamma(\mathcal{V})$.*

- *$[X, Y]$ is projectable and π-related to $[\bar{X}, \bar{Y}]$.*
- *$[X, Z]$ is vertical.*

Proof. We refer the reader to [35] for the straightforward proofs of the lemmas. □

Remark 4.3.4. While basic fields are only sensible for submersions, we recall that foliations are always locally submersions and so the notion of basic fields can also sometimes be useful for local computations on general foliations.

Let $R^{\mathbb{B}}$ denote the Riemann curvature tensor on $\mathbb{B}$ for the Levi-Civita connection, and $R_{\mathcal{H}}$ as in lemma 3.2.5. We will focus on the relationship between these, for which the notion of basic fields is a useful tool.

Lemma 4.3.5. *Let X, Y, Z be basic vector fields on $\mathbb{M}$, and denote by $R^{\mathbb{B}}$ the Riemann curvature tensor on $\mathbb{B}$. Then $R_{\mathcal{H}}(X, Y)Z$ is basic and is π-related to $R^{\mathbb{B}}(\bar{X}, \bar{Y})\bar{Z}$.*

Proof. Observe that on $\mathbb{M}$,

$$\nabla_{\mathrm{pr}_{\mathcal{V}}[X,Y]} Z = -\nabla_{T(X,Y)} Z = -\,\mathrm{pr}_{\mathcal{H}}[T(X, Y), Z] = 0$$

where we use properties of the Bott connection for the first two equalities and lemma 4.3.3 for the last. As a consequence,

$$R_{\mathcal{H}}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\mathrm{pr}_{\mathcal{H}}[X,Y]} Z$$

We now establish the following

Lemma 4.3.6. *For basic vector fields X, Y on $\mathbb{M}$,*

- $\nabla_X Y$ *is basic and π-related to* $\nabla^{\mathbb{B}}_{\bar{X}} \bar{Y}$
- $\mathrm{pr}_{\mathcal{H}}[X,Y]$ *is basic and π-related to* $[\bar{X}, \bar{Y}]$

Proof. First, that $\nabla^{\mathbb{B}}_{\bar{X}} \bar{Y}$ is π-related to $\mathrm{pr}_{\mathcal{H}}(\nabla^{\mathbb{M}}_X Y)$ follows from writing the lift of $\nabla^{\mathbb{B}}_{\bar{X}} \bar{Y}$ in a local coordinate chart using the Christoffel symbols. Then since X and Y are horizontal, $\nabla_X Y = \mathrm{pr}_{\mathcal{H}}(\nabla^{\mathbb{M}}_X Y)$ follows from the properties of the Bott connection and the first claim is established.

The second claim follows directly from lemma 4.3.3. $\square$

Applying the lemma, we compute on $\mathbb{B}$ that

$$\begin{aligned} R^{\mathbb{B}}(\bar{X},\bar{Y})\bar{Z} &= \nabla^{\mathbb{B}}_{\bar{X}} \nabla^{\mathbb{B}}_{\bar{Y}} \bar{Z} - \nabla^{\mathbb{B}}_{\bar{Y}} \nabla^{\mathbb{B}}_{\bar{X}} \bar{Z} - \nabla^{\mathbb{B}}_{[\bar{X},\bar{Y}]} \bar{Z} \\ &= \overline{\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\mathrm{pr}_{\mathcal{H}}[X,Y]} Z}. \end{aligned}$$

and the lemma follows. $\square$

Denote by $\overline{\mathbf{Hol}}(\mathbb{B}, \bar{p})$ the holonomy group of $\mathbb{B}$ at $\bar{p} \in \mathbb{B}$ for the Levi-Civita connection, and by $\overline{\mathfrak{hol}}(\mathbb{B}, \bar{p})$ its Lie algebra. These are again isomorphic for all $\bar{p} \in \mathbb{B}$, and so we can write $\overline{\mathbf{Hol}}(\mathbb{B})$ and $\overline{\mathfrak{hol}}(\mathbb{B})$.

In the following, we consistently denote by $\gamma\colon [0,T] \to \mathbb{M}$ piecewise-smooth horizontal curves, set $p = \gamma(0)$, and their ∇-parallel transport by $\tau_{t,\gamma}\colon T_p\mathbb{M} \to T_{\gamma(t)}\mathbb{M}$. Similarly, we denote by $\bar{\gamma}\colon [0,T] \to \mathbb{B}$ piecewise-smooth curves, $\bar{p} = \bar{\gamma}(0)$, and their $\nabla^{\mathbb{B}}$-parallel transport by $\bar{\tau}_{t,\bar{\gamma}}\colon T_{\bar{p}}\mathbb{B} \to T_{\bar{\gamma}(t)}\mathbb{B}$. We will also write $\tau_\gamma = \tau_{T,\gamma}$ and $\bar{\tau}_{\bar{\gamma}} = \bar{\tau}_{T,\bar{\gamma}}$ when convenient.

Theorem 4.3.7. *For an H-type submersion* $(\mathbb{M}, \mathcal{H}, g, \pi)$,

$$\mathbf{Hol}^0(\mathcal{H}) \cong \overline{\mathbf{Hol}}^0(\mathbb{B})$$

Proof. For $p \in \mathbb{M}$, we want to relate the Lie algebras $\mathfrak{hol}(\mathcal{H}, p)$ and $\overline{\mathfrak{hol}}(\mathbb{B}, \pi(p))$. Using the Ambrose-Singer theorem and our theorem 4.2.3, this is reduced to studying the parallel transports $\tau_\gamma, \bar{\tau}_{\bar{\gamma}}$ and curvature tensors $R_{\mathcal{H}}$ and $R^{\mathbb{B}}$.

We need the following

Lemma 4.3.8. *Suppose γ is a piecewise-C^1 curve on $\mathbb{M}$ and let $\bar{\gamma}$ be the piecewise-C^1 curve on $\mathbb{B}$ determined by $\bar{\gamma}(t) = \pi(\gamma(t))$. Let $u \in \mathcal{H}_p$ and set $\bar{u} = d_p\pi(u) \in T_{\bar{p}}\mathbb{B}$. Then for all $t \in [0,T]$,*

$$d_{\gamma(t)}\pi(\tau_{t,\gamma}(u)) = \bar{\tau}_{t,\bar{\gamma}}(\bar{u}).$$

Proof. Let $Y(t) = \tau_{t,\gamma}(u)$ and $\bar{Y}\colon [0,T] \to T\mathbb{B}$ be the pushforward

$$\bar{Y}(t) = d_{\gamma(t)}\pi(Y(t)) \in T_{\bar{\gamma}(t)}\mathbb{B}.$$

Then

$$D_t^{\mathbb{B}}\bar{Y} = d_{\gamma(t)}\pi(D_tY) = 0$$

since Y is the ∇-parallel transport of u (where $D^{\mathbb{B}}, D^{\mathbb{M}}, D$ are the respective covariant derivatives along γ, which do not depend on the choice of extension of $Y, \bar{Y}, \gamma'$, and $\bar{\gamma}'$). Thus $\bar{Y}$ is the $\nabla^{\mathbb{B}}$-parallel transport of $\bar{u}$ along $\bar{\gamma}$, and the lemma follows from the uniqueness of parallel transport. □

Now fix $p \in \mathbb{M}$; there is a one-to-one correspondence between piecewise-smooth horizontal curves γ on $\mathbb{M}$ with $\gamma(0) = p$ and piecewise-smooth curves $\bar{\gamma}$ on $\mathbb{B}$ with $\bar{\gamma}(0) = \pi(p)$ determined by the projection. For any such pair of curves and π-related pairs of vectors $x, y, z \in \mathcal{H}_p, \bar{x}, \bar{y}, \bar{z} \in T_{\pi(p)}\mathbb{B}$ an application of lemma 4.3.5 and lemma 4.3.8 give us that

$$d_p\pi(\tau_\gamma^{-1} R_{\mathcal{H}}(\tau_\gamma(x), \tau_\gamma(y))\tau_\gamma z) = \bar{\tau}_{\bar{\gamma}}^{-1} R^{\mathbb{B}}(\bar{\tau}_{\bar{\gamma}}(\bar{x}), \bar{\tau}_{\bar{\gamma}}(\bar{y}))\bar{\tau}_{\bar{\gamma}}\bar{z}$$

and so by theorem 4.2.3

$$\mathfrak{hol}(\mathcal{H}, p) \cong \overline{\mathfrak{hol}}(\mathbb{B}, \pi(p))$$

and the theorem follows. □

Recalling theorem 3.2.12, we have a complete list of the possible horizontal holonomies of H-type submersions.

4.3.2 Holonomy of H-type foliations with horizontally parallel Clifford structure

We now consider the more general setting of foliations that are not globally submersions. In view of theorem 4.2.3, to study the horizontal holonomy group $\mathbf{Hol}(\mathcal{H})$ the study of the symmetries of the horizontal endomorphisms $R_{\mathcal{H}}$ will be of paramount importance. Given an horizontally parallel Clifford structure, we recall the useful lemma 3.4.10.

Lemma 4.3.9. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with* $m = \operatorname{rank}(\mathcal{V}) \geq 2$ *and parallel horizontal Clifford structure* $\Psi(u, v) = -\kappa(u \cdot v + \langle u, v \rangle)$*. Then*

$$[R_{\mathcal{H}}(u, v), J_z] = \kappa \sum_{j=1, j \neq i}^{m} \Big(\langle J_j u, v \rangle J_{ij} - \langle J_{ij} u, v \rangle J_j \Big).$$

for all $u, v \in \mathcal{H}_p, z \in \mathcal{V}_p$*.*

In particular, we consider the cases when the right hand side has particularly nice structure.

Corollary 4.3.10. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with parallel horizontal Clifford structure.*

- *If* $\kappa = 0$ *then* $R_{\mathcal{H}}(u, v)$ *commutes with* J_z*.*
- *If* $\mathbb{M}$ *is quaternionic, that is if* $m = 3$ *and* $\mathfrak{a}(p) = \{J_z, z \in \mathcal{V}\} \cong \mathfrak{so}(3)$*, then* $R_{\mathcal{H}}(u, v)$ *will preserve* $\mathfrak{a}(p)$*.*

From this, the following structural theorem suggested by theorem 3.4.7 will follow.

Theorem 4.3.11. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with parallel horizontal Clifford structure, and set* $n = \operatorname{rank}(\mathcal{H}), m = \operatorname{rank}(\mathcal{V})$*.*

(a) *If* $m = 1$*, then* $\mathbf{Hol}^0(\mathcal{H})$ *is isomorphic to a subgroup of* $\mathbf{U}(n/2)$*.*

(b) *If* $m \geq 2$ *and* $\kappa = 0$*, then* $\mathbf{Hol}^0(\mathcal{H})$ *is isomorphic to a subgroup of* $\mathbf{Sp}(n/4)$*.*

(c) *If* $m = 3$ *and and the maps* $J_z, z \in \mathcal{V}$ *form a Lie algebra under commutation at every point, then* $\mathbf{Hol}^0(\mathcal{H})$ *is isomorphic to a subgroup of* $\mathbf{Sp}(1)\mathbf{Sp}(n/4)$

Proof. Let $p \in \mathbb{M}$ be a chosen base point, $\gamma \in \mathcal{C}^0_p$. The strategy of the proof is to explicitly construct appropriate complex and symplectic forms.

(a) If $m = 1$, then $\mathbf{Hol}(\mathcal{V}) = 1$, so $\tau_\gamma z = z$ for any $z \in \mathcal{V}_p$ and since J is also parallel we have $\tau_\gamma J_z = J_z \tau_\gamma$. Fix a unit vector $z \in \mathcal{V}_p$, and then $J = J_z = T^\flat(\cdot,\cdot,z)^{\sharp_1}$ will be a complex structure on $T_p\mathbb{M}$. We can now define a complex inner product on $\mathcal{H}_p$ by

$$\langle u, w\rangle_{p,\mathbb{C}} = g_{p,\mathcal{H}}(u, w + Jw)$$

We will then have $\langle \tau_\gamma u, \tau_\gamma w\rangle_{p,\mathbb{C}} = \langle u, w\rangle_{p,\mathbb{C}}$, which implies $\tau_\gamma \in \mathfrak{U}(n/2)$.

Since $\operatorname{rank} \mathcal{V} = 1$ it must be that $R_\mathcal{H}(u, v)$ will commute with J. The result follows from theorem 4.2.3.

(b) We note that since $\kappa = 0$, we have that J is parallel and that $R_\mathcal{V} = 0$, so $\mathbf{Hol}(\mathcal{V}) = 1$. Fixing orthogonal unit vectors $z_1, z_2 \in \mathcal{V}_p$ we can define $J_k = J_{z_k}$ for $k = 1, 2$. We note that $J_1 J_2 = -J_2 J_1$ and that $\tau_\gamma J_k = J_k \tau_\gamma$. We use J_1 as a complex structure on $\mathcal{H}_p$ and repeat the argument in (a) to show that, with respect to the appropriate choice of basis, $\tau_\gamma \in \mathbf{U}(n/2)$. Furthermore,

$$\omega(u, v) = \langle J_1 u, v\rangle_\mathcal{H} + i\langle J_1 J_2 u, v\rangle_\mathcal{H}, \qquad u, v \in \mathcal{H}_p,$$

is a complex symplectic form on $\mathcal{H}_p$, meaning that relative to the same basis $\tau_\gamma \in \mathbf{Sp}(n/4)$.

Since $\kappa = 0$ we have that $R_\mathcal{H}(u, v)$ will commute with any J_k by corollary 4.3.10. The result follows from theorem 4.2.3.

(c) From our assumption, it follows that the subbundle $\{J_z \in \operatorname{End}(\mathcal{H}_p) \colon z \in \mathcal{V}\}$ is preserved under parallel transport. Hence there is some map $Q \in \mathrm{End}(\mathcal{V}_p)$ such that $\tau_\gamma J_z v = J_{Qz}\tau_\gamma v$. Furthermore, we have that for any unit vector $v \in \mathcal{H}_p$,

$$g_{p,\mathcal{V}}(z_1, z_2) = g_{p,\mathcal{H}}(\tau_\gamma J_{z_1} v, \tau_\gamma J_{z_2} v) = g_{p,\mathcal{H}}(J_{Qz_1}\tau_\gamma v, J_{Qz_2}\tau_\gamma v) = g_{p,\mathcal{V}}(Qz_1, Qz_2)$$

so Q is an (orientation preserving) isometry as well. Since $\mathfrak{a}(p)$ is a subalgebra of the skew-symmetric endomorphisms isomorphic to $\mathfrak{sp}(1)$, there exists a unique element $A \in \exp\{J_z \colon z \in \mathcal{V}_p\}$ such that

$$AJ_zA^{-1}z = \tau_\gamma J_z \tau_\gamma^{-1} = J_{Qz}.$$

It follows that $\tilde{\tau_\gamma} := A^{-1}\tau_\gamma$ satisfies $\tilde{\tau_\gamma} J_z = J_z \tilde{\tau_\gamma}$. By similar arguments as in (b), it follows that $\tilde{\tau_\gamma}$ can be made unitary and symplectic.

By corollary 4.3.10, $R_{\mathcal{H}}(u, v)$ will preserve $\mathfrak{a}(p)$. The result follows from theorem 4.2.3.

□

Remark 4.3.12. The outstanding cases occur for $m \geq 2, \kappa \neq 0$, excluding the quaternionic case for $m = 3$. Equivalently, these are the cases for which $\mathfrak{a}(p) \cong \{J_{z_1}, [J_{z_1}, J_{z_2}] \colon z_1, z_2 \in \mathcal{V}_p\} \cong \mathfrak{so}(m+1)$.

Chapter 5

Sub-Riemannian Comparison Theorems on H-Type Foliations

Much of the content of this chapter overlaps with a paper coauthored with Baudoin, Grong, and Rizzi in 2019. For the complete proofs of those results we will refer to the original paper [25].

In this chapter we consider the sub-Riemannian structure of H-type foliations as the limit of Riemannian metrics. In particular, we study the distance function and are able to recover purely sub-Riemannian comparison results that classically rely on Ricci curvature bounds that cannot naturally exist as limits of the Riemannian structure.

5.1 Riemannian comparison theorems

One well-established approach to Riemannian geometry is through comparison principles; one computes precisely some quantity of interest on model spaces then determines under what conditions a manifold can be compared to the model spaces, and thereby estimates can be established. For example, we can consider bounds on the distance function and its Laplacian.

Fix a point p in a Riemannian manifold $(\mathbb{M}, g)$ equipped with the Levi-Civita connection ∇^g. We can define the distance function $r_p \colon \mathbb{M} \to \mathbb{R}$ by

$$r_p(q) = d_g(p, q)$$

which is smooth outside of the cut locus $\mathbf{Cut}_g(p)$. The properties of this function are of great interest. An upper bound on it establishes a diameter bound, since we can see that

$$\operatorname{diam}(\mathbb{M}, d_g) = \sup_{p,q \in \mathbb{M}} r_p(q).$$

For example, given a Ricci curvature condition we have a comparison result on the diameter as follows.

Theorem 5.1.1 (Bonnet-Myers [91])**.** *Let* $(\mathbb{M}, g)$ *be an* n*-dimensional Riemannian manifold equipped with the Levi-Civita connection. Then if there exists a constant* $\rho > 0$ *such that*

$$\operatorname{Ric} \geq (n-1)\rho$$

then the diameter bound

$$\operatorname{diam}(\mathbb{M}, d_g) \leq \frac{\pi}{\sqrt{\rho}}$$

holds and the manifold is compact with finite fundamental group.

This theorem is established by consideration of the Hessian of the distance function, and comparison with the model space of positive curvature, the sphere, where it holds that

$$\text{Ric} = (n-1)\rho$$
$$\text{diam}(S^n, d_g) = \frac{\pi}{\sqrt{\rho}}$$

There is a rigidity theorem stating that the sphere uniquely expresses this property.

Theorem 5.1.2 (Cheng [50]). *Suppose* $(\mathbb{M}, g)$ *is an* n*-dimensional Riemannian manifold equipped with the Levi-Civita connection and such that* $\text{Ric} \geq (n-1)\rho$*. If there exist points* $p, q \in \mathbb{M}$ *such that*

$$d_g(p,q) = \frac{\pi}{\sqrt{\rho}}$$

then $\mathbb{M}$ *is isometric to the* n*-dimensional sphere* S^n*.*

Under the same conditions on the Ricci curvature, there is a classical estimate for the Laplacian of the distance function. We define for later convenience the following function.

Definition 5.1.3 (Riemannian comparison function).

$$F_{\text{Riem}}(r,k) = \begin{cases} \sqrt{k}\cot\sqrt{k}r & \text{if } k > 0 \\ \frac{1}{r} & \text{if } k = 0 \\ \sqrt{|k|}\coth\sqrt{|k|}r & \text{if } k < 0 \end{cases}$$

In terms of this function we have the following comparison result.

Theorem 5.1.4 (Laplacian comparison theorem). *Suppose* $(\mathbb{M}, g)$ *is an* n*-dimensional Riemannian manifold equipped with Levi-Civita connection and that there exists* $\rho \in \mathbb{R}$

such that $\mathrm{Ric} \geq (n-1)\rho$. *Then*

$$\nabla r \leq F_{\mathrm{Riem}}(r, \rho).$$

The proof follows from establishing bounds on the Hessian evaluated on Jacobi fields, and follows as a solution of a Riccati equation; it can be considered to be a special case of the Rauch comparison theorem [99, 44, 34]. See [83, 96] for a modern presentation.

5.2 H-type foliations as limits of Riemannian Manifolds

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. In previous sections we've seen that we can study the sub-Riemannian geometry by investigating the J map, relating properties of $\mathcal{V}$ and especially $\mathbf{Cl}(\mathcal{V})$ to that of $\mathcal{H}$. There is, however, a stronger sense in which we can construct the purely sub-Riemannian structure $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ as a limit. Recall the notion of penalty metric from section 1.2.1.

Definition 5.2.1. Let $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ be a sub-Riemannian manifold, and let $(\mathbb{M}, g)$ be a Riemannian manifold such that $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$ is an extension of $g_{\mathcal{H}}$. We define the associated penalty metric

$$g_{\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$$

In the particular case of H-type foliations we have the Gromov-Hausdorff convergence theorem 1.2.2, but also a stronger statement that is essential to our approach to sub-Riemannian comparison theorems.

Theorem 5.2.2. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation equipped with the penalty metric g_ε. The convergence $d_\varepsilon \xrightarrow{\varepsilon \to 0^+} d_0 = d_{cc}$ is uniform on compact sets.*

Proof. This is subtle, and one of the motivating reasons for the study of H-type foliations. We refer to [25] for details of this result, as well as [1, 102] for further discussion of the convergence of distance functions in this sense. □

5.2.1 The comparison principle for H-type foliations

We begin by observing that the Bott connection is extremely useful for the study of penalty metrics.

Lemma 5.2.3. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation equipped with the Bott connection ∇, and let $g_\varepsilon = g_\mathcal{H} \oplus \frac{1}{\varepsilon} g_\mathcal{V}$ be the associated penalty metric. Then ∇ is g_ε-metric for all $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned}
(\nabla_X g_\varepsilon)(Y, Z) &= X \cdot g_\varepsilon(Y, Z) - g_\varepsilon(\nabla_X Y, Z) - g_\varepsilon(Y, \nabla_X Z) \\
&= X \cdot \left(g_\mathcal{H}(Y, Z) + \frac{1}{\varepsilon} g_\mathcal{V}(Y, Z) \right) \\
&\quad - \left(g_\mathcal{H}(\nabla_X Y, Z) + \frac{1}{\varepsilon} g_\mathcal{V}(\nabla_X Y, Z) \right) \\
&\quad - \left(g_\mathcal{H}(Y, \nabla_X Z) + \frac{1}{\varepsilon} g_\mathcal{V}(Y, \nabla_X Z) \right) \\
&= X \cdot g(\operatorname{pr}_\mathcal{H} Y, \operatorname{pr}_\mathcal{H} Z) - g(\nabla_X \operatorname{pr}_\mathcal{H} Y, \operatorname{pr}_\mathcal{H} Z) - g(\operatorname{pr}_\mathcal{H} Y, \nabla_X \operatorname{pr}_\mathcal{H} Z) \\
&\quad + \frac{1}{\varepsilon} \left(X \cdot g(\operatorname{pr}_\mathcal{V} Y, \operatorname{pr}_\mathcal{V} Z) - g(\nabla_X \operatorname{pr}_\mathcal{V} Y, \operatorname{pr}_\mathcal{V} Z) - g(\operatorname{pr}_\mathcal{V} Y, \nabla_X \operatorname{pr}_\mathcal{V} Z) \right) \\
&= (\nabla_X g)(\operatorname{pr}_\mathcal{H} Y, \operatorname{pr}_\mathcal{H} Z) + \frac{1}{\varepsilon} (\nabla_X g)(\operatorname{pr}_\mathcal{V} Y, \operatorname{pr}_\mathcal{V} Z) = 0
\end{aligned}$$

where we use in an essential way that for the Bott connection $\mathrm{pr}_{\mathrm{E}} \nabla_X Y = \nabla_X \mathrm{pr}_{\mathrm{E}} Y$.

□

We will be studying the Jacobi equation, and as a consequence it will be desirable to have a metric adjoint connection as in section 2.3. Therefore we define for $\varepsilon > 0$ the map $J^\varepsilon = \frac{1}{\varepsilon} J$ and write

$$\widehat{\nabla^\varepsilon}_X Y = \nabla_X Y + J^\varepsilon_X Y, \tag{5.2.1}$$

following the notation of [23]. Its adjoint is then given by

$$\nabla^\varepsilon_X Y = \nabla_X Y - T(X,Y) + J^\varepsilon_Y X.$$

We stress that both $\widehat{\nabla^\varepsilon}$ and ∇^ε are g_ε-metric for any $\varepsilon > 0$.

Let $\widehat{R^\varepsilon}(X,Y) = \widehat{\nabla^\varepsilon}_X \widehat{\nabla^\varepsilon}_Y - \widehat{\nabla^\varepsilon}_Y \widehat{\nabla^\varepsilon}_X - \widehat{\nabla^\varepsilon}_{[X,Y]}$ be the Riemann curvature tensor of $\widehat{\nabla^\varepsilon}$. By section 2.3.2 we see that a vector field W along a g_ε-geodesic γ is a Jacobi field if and only if it satisfies the Jacobi equation $\mathcal{Z}(W) = 0$ for the Jacobi operator

$$\mathcal{Z}(W) = \widehat{\nabla^\varepsilon}_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W + \widehat{R^\varepsilon}(W, \dot\gamma)\dot\gamma$$

We can establish a comparison principle as in section 2.3.2 for H-type foliations in terms of the penalty metric g_ε.

Theorem 5.2.4 ([25, Theorem 2.11])**.** *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation.*

- *Let $g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$ be the associated penalty metric, and fix $\varepsilon > 0$.*
- *Choose $x \in \mathbb{M}$ and $y \notin \mathbf{Cut}_\varepsilon(x)$, and let $\gamma_\varepsilon : [0, r_\varepsilon] \to \mathbb{M}$ be the unique g_ε-geodesic, parametrized with unit speed, joining x with y.*

- *For $\ell \in \mathbb{N}$, let $W_1, \dots, W_\ell$ be vector fields along γ_ε and g_ε-orthogonal to $\dot{\gamma}_\varepsilon$ such that*

$$\sum_{i=1}^{\ell} \int_0^r g_\varepsilon(\mathcal{Z}(W_i), W_i)\, dt \geq 0. \tag{5.2.2}$$

Then, at $y = \gamma_\varepsilon(r_\varepsilon)$, it holds

$$\sum_{i=1}^{\ell} \mathrm{Hess}^{\widehat{\nabla^\varepsilon}}(r_\varepsilon)(W_i, W_i) \leq \sum_{i=1}^{\ell} g_\varepsilon(W_i(r_\varepsilon), \widehat{\nabla^\varepsilon}_{\dot{\gamma}_\varepsilon} W_i(r_\varepsilon)), \tag{5.2.3}$$

where equality holds if and only if $W_1, \dots, W_\ell$ are Jacobi fields for the metric g_ε.

Proof. This is simply the special case of theorem 2.3.13 where the metric is parameterized by $\varepsilon > 0$. □

Lemma 5.2.5 ([25, Lemma 2.12])**.** *If u is sufficiently regular, then one has*

$$\mathrm{Hess}^{\widehat{\nabla^\varepsilon}}(u)(W, W) = \mathrm{Hess}^{\nabla}(u)(W, W) + \frac{1}{\varepsilon} g(J_{\mathrm{pr}_{\mathcal{V}} W} du^\sharp, \mathrm{pr}_{\mathcal{H}} W)$$

Proof. For a g-metric connection ∇ there is a known expression for the Hessian,

$$\mathrm{Hess}^{\nabla}(u)(X, Y) = (\nabla_X du)Y = g(\nabla_X du^\sharp, Y).$$

Applying lemma 5.2.3 and the defining equation eq. (5.2.1) we can compute

$$\begin{aligned} \mathrm{Hess}^{\widehat{\nabla^\varepsilon}}(u)(W, W) &= g_\varepsilon((\nabla_W + J^\varepsilon_W) du^\sharp, W) \\ &= \mathrm{Hess}^{\nabla}(u)(W, W) + g(J^\varepsilon_W du^\sharp, W) \end{aligned}$$

and the result follows from properties of the $J^\varepsilon = \frac{1}{\varepsilon} J$ map. □

Corollary 5.2.6. *If the W_i in theorem 5.2.4 are horizontal at $y = \gamma(r_\varepsilon)$, then the Hessian in* (5.2.3) *can be computed equivalently using the Hessian for ∇; that is*

$$\sum_{i=1}^{\ell} \mathrm{Hess}^{\nabla}(r_\varepsilon)(W_i, W_i) \le \sum_{i=1}^{\ell} g_\varepsilon(W_i(r_\varepsilon), \widehat{\nabla^\varepsilon}_{\dot\gamma_\varepsilon} W_i(r_\varepsilon)),$$

Remark 5.2.7. This corollary is essential, as it implies that $\mathrm{Hess}^{\nabla}(r_\varepsilon)$ can be controlled by consideration of fields satisfying a differential equation in ε. The main results will follow from establishing formulas uniform in $\varepsilon > 0$ and then taking the sub-Riemannian limit.

In order to verify condition (5.2.2) of theorem 5.2.4 it will be useful to write explicitly the Jacobi operator in terms of the Bott connection and its curvature. In the next lemma we do this for the case of H-type foliations with horizontally parallel torsion.

Lemma 5.2.8 ([25, Lemma 2.14])**.** *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation that admits a parallel horizontal Clifford structure with constant κ. Let W be a vector field along a g_ε-geodesic γ with $\varepsilon > 0$. Let $W_\perp$ be the g_ε-orthogonal projection of W on the orthogonal complement of $\dot\gamma$. Then*

$$\begin{aligned}\mathcal{Z}(W) = {}& \widehat{\nabla^\varepsilon}_{\dot\gamma}\widehat{\nabla^\varepsilon}_{\dot\gamma} W - J^\varepsilon_{\dot\gamma}\widehat{\nabla^\varepsilon}_{\dot\gamma} W + J^\varepsilon_{\widehat{\nabla^\varepsilon}_{\dot\gamma} W}\dot\gamma + \kappa J_{\dot\gamma} J^\varepsilon_{(W_\mathcal{V})_\perp}\dot\gamma + J^\varepsilon_{T(W,\dot\gamma)}\dot\gamma + R_\mathcal{H}(W,\dot\gamma)\dot\gamma \\ & + \widehat{\nabla^\varepsilon}_{\dot\gamma}(T(W,\dot\gamma)) + \kappa(T(J_{\dot\gamma} W, \dot\gamma) + \langle W, \dot\gamma_\mathcal{H}\rangle \dot\gamma_\mathcal{V}) + \kappa^2 \|\dot\gamma_\mathcal{V}\|^2 (W_\mathcal{V})_\perp.\end{aligned}$$

Proof. We can write the Jacobi operator by expanding the adjoint $\nabla^\varepsilon = \widehat{\nabla^\varepsilon} - \widehat{T^\varepsilon}$ as

$$\mathcal{Z}(W) = \widehat{\nabla^\varepsilon}_{\dot\gamma}(\widehat{\nabla^\varepsilon}_{\dot\gamma} W - \widehat{T^\varepsilon}(\dot\gamma, W)) + \widehat{R^\varepsilon}(W,\dot\gamma)\dot\gamma.$$

The proof follows by explicit computation of the horizontal and vertical part of the above equation. We refer to [25] for the details. □

5.3 Comparison theorems for H-type foliations with parallel horizontal Clifford Structure

In this section, we will use theorem 5.2.4 to obtain bounds on $\operatorname{Hess}^{\widehat{\nabla}^\varepsilon}(r_\varepsilon)$ and therefore on $\operatorname{Hess}^{\nabla}(r_\varepsilon)$ by corollary 5.2.6. The approach is related to the proof of theorem 5.1.4 but requires some subtlety. In particular, we will need a decomposition of $\mathcal{H}$ that distinguishes between directions associated to the geodesic by the J map and those which are not.

5.3.1 The Splitting

Definition 5.3.1. For $Y \in \Gamma(T\mathbb{M}), \operatorname{pr}_{\mathcal{H}} Y \neq 0$ we call the orthogonal splitting

$$\mathcal{H} = \mathcal{H}_{\mathrm{Sas}}(Y) \oplus \mathcal{H}_{\mathrm{Riem}}(Y) \oplus \operatorname{span}(\operatorname{pr}_{\mathcal{H}} Y)$$

the <u>canonical splitting along Y</u>, where

$$\begin{aligned}\mathcal{H}_{\mathrm{Sas}}(Y) &= \{J_Z Y \colon Z \in \mathcal{V}\}\\ \mathcal{H}_{\mathrm{Riem}}(Y) &= \{X \in \mathcal{H} \colon X \perp \big(\mathcal{H}_{\mathrm{Sas}}(Y) \oplus \operatorname{span}(\operatorname{pr}_{\mathcal{H}} Y)\big)\}\end{aligned}$$

This splitting will be important, and is natural for the connection $\widehat{\nabla}^\varepsilon$ in the following sense.

Lemma 5.3.2 ([25, Proposition 3.2])**.** *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with par-*

allel horizontal Clifford structure and satisfying the J^2 condition. Let Y be a $\widehat{\nabla^{\varepsilon}}_Y$-parallel vector field. Then each sub-bundle composing the canonical splitting along Y is preserved by $\widehat{\nabla^{\varepsilon}}_Y$-parallel transport. We will say that the splitting is $\widehat{\nabla^{\varepsilon}}_Y$-parallel, or just parallel.

Proof. It's sufficient to show that $\mathcal{H}_{\mathrm{Sas}}(Y)$ is $\widehat{\nabla^{\varepsilon}}_Y$-parallel. For $X = J_Z Y \in \mathcal{H}_{\mathrm{Sas}}(Y)$ we can write

$$\widehat{\nabla^{\varepsilon}}_Y X = J_{\Psi(Y,Z)} Y + J_{\nabla_Y Z} Y + \frac{1}{\varepsilon} J_Y J_Z Y.$$

By choosing Z to be $\widehat{\nabla^{\varepsilon}}_Y$-parallel and splitting into the cases $Z \propto \mathrm{pr}_{\mathcal{V}} Y$ and $Z \perp \mathrm{pr}_{\mathcal{V}} Y$ it follows that there exists a basis of $\mathcal{H}_{\mathrm{Sas}}(Y)$ closed under $\widehat{\nabla^{\varepsilon}}$-covariant derivation, completing the proof. The details can be found in [25, proposition 3.2]. □

In particular, we will be interested in the case $\gamma\colon [0,T] \to \mathbb{M}$ is a geodesic, along which we will study the frame determined by the canonical splitting along $\dot{\gamma}$.

Remark 5.3.3. In the case that $(\mathbb{M}, \mathcal{H}, g)$ does not satisfy the J^2 condition, it is necessary to further refine the splitting. We first orthogonally split $\mathcal{V} = \mathcal{V}_{\mathrm{Sas}}(Y) \oplus \mathcal{V}_{\mathrm{Htype}}(Y)$ as

$$\mathcal{V}_{\mathrm{Sas}}(Y) = \{Z \in \mathcal{V} \colon J_Y J_Z Y \subseteq J_{\mathcal{V}}(Y) \oplus \mathrm{span}(\mathrm{pr}_{\mathcal{H}} Y)\}$$
$$\mathcal{V}_{\mathrm{Htype}}(Y) = \{Z \in \mathcal{V} \colon J_Y J_Z Y \perp \big(J_{\mathcal{V}} \oplus \mathrm{span}(\mathrm{pr}_{\mathcal{H}} Y)\big), J_Y J_Z Y \neq 0\}$$

where $\mathcal{V}_{\mathrm{Htype}}(Y)$ is motivated as being the space that precisely captures the vertical vectors $Z \in \mathcal{V}$ for which the way we utilize the J^2 condition in the following sections fails. We then analogously split $\mathcal{H} = \mathcal{H}_{\mathrm{Sas}}(Y) \oplus \mathcal{H}_{\mathrm{Htype}}(Y) \oplus \mathcal{H}_{\mathrm{Sas}}(Y) \oplus \mathrm{span}(\mathrm{pr}_{\mathcal{H}} Y)$

as

$$\mathcal{H}_{\text{Sas}}(Y) = \{J_Z Y : Z \in \mathcal{V}_{\text{Sas}}(Y)\}$$
$$\mathcal{H}_{\text{Htype}}(Y) = \{J_Z Y, J_Y J_Z Y : Z \in \mathcal{V}_{\text{Htype}}\}$$
$$\mathcal{H}_{\text{Riem}}(Y) = \{X \in \mathcal{H} : X \perp \big(\mathcal{H}_{\text{Htype}} \oplus \mathcal{H}_{\text{Sas}}(Y) \oplus \operatorname{span}(\operatorname{pr}_{\mathcal{H}} Y)\big)\}.$$

The comparison theorems in the following sections then have to be slightly modified, in particular to include a new theorem on $\mathcal{H}_{\text{Htype}}$, but the overall conclusion is very similar.

Notice, however, that if $\operatorname{pr}_{\mathcal{V}} Y = 0$ then $\mathcal{V}_{\text{Htype}}(Y) = \mathcal{H}_{\text{Htype}}(Y) = \emptyset$ and the splitting reduces to the canonical one. It will hold that in the sub-Riemannian limit $\varepsilon \to 0^+$ the comparisons converge uniformly, and since our primary motivation is the study of the sub-Riemannian geometry (where $Y = \dot\gamma \in \mathcal{H}$) we will not give further details, but refer to section 3.7.1 through 3.7.3 of [25].

5.3.2 Comparison theorems along the canonical splitting

As in the Riemannian setting, the goal is to establish a comparison theorem relating the curvature of $\mathbb{M}$ to the Hessian of the distance function $r(y) = d(x, y)$ for some fixed $x \in \mathbb{M}$. Note that in our setting the distance function $r_\varepsilon(y) := d_\varepsilon(x, y)$ depends on the choice of Riemannian metric g_ε, $\varepsilon > 0$. For vectors $X \in \mathcal{H}$ we will establish comparison theorems for $\operatorname{Hess}(r_\varepsilon)(X, X)$ by distinguishing between X in each of the components of the canonical decomposition along $\dot\gamma$, where γ is the length-minimizing geodesic connecting x to $y \notin \mathbf{Cut}_\varepsilon(x)$. We establish some notation and make the following remarks to keep the statement of the theorem as simple as possible.

For a unit speed g_ε-geodesic $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ containing no conjugate points joining

$x = \gamma(0)$ to $y = \gamma(r_\varepsilon) \notin \mathbf{Cut}_\varepsilon(x)$ it holds that $d_\varepsilon(x, \gamma(t)) = t$. In particular, $r_\varepsilon(y) = d_\varepsilon(x, y)$ and $\dot\gamma = \nabla^{g_\varepsilon} r_\varepsilon$. We can write

$$\dot\gamma = \mathrm{pr}_{\mathcal{H}}\, \dot\gamma + \mathrm{pr}_{\mathcal{V}}\, \dot\gamma = \nabla_{\mathcal{H}} r_\varepsilon + \varepsilon \nabla_{\mathcal{V}} r_\varepsilon.$$

We will also denote by $h = \|\nabla_{\mathcal{H}} r_\varepsilon\|$ and $v = \|\nabla_{\mathcal{V}} r_\varepsilon\|$ as measured by the $g_1 = g$ metric. Because γ is unit speed for the g_ε-metric we have the eikonal equation

$$\|\nabla^{g_\varepsilon} r_\varepsilon\|_\varepsilon^2 = \|\dot\gamma\|_\varepsilon^2 = h^2 + \varepsilon v^2 = 1.$$

See [54, 57] for a thorough discussion of geodesics and the eikonal equation. We note that as $\varepsilon \to 0^+$ (the sub-Riemannian limit) we will have $h \to 1, v \to 0$, or equivalently all geodesics become horizontal.

We recall the Riemannian comparison function definition 5.1.3, and establish

Definition 5.3.4 (Sasakian comparison function)**.**

$$F_{\mathrm{Sas}}(r,k) = \begin{cases} \frac{\sqrt{k}(\sin\sqrt{k}r - \sqrt{k}r\cos\sqrt{k}r)}{2-2\cos\sqrt{k}r - \sqrt{k}r\sin\sqrt{k}r} & \text{if } k > 0, \\ \frac{4}{r} & \text{if } k = 0, \\ \frac{\sqrt{|k|}(\sqrt{|k|}r\cosh\sqrt{|k|}r - \sinh\sqrt{|k|}r)}{2-2\cosh\sqrt{|k|}r + \sqrt{|k|}r\sinh\sqrt{|k|}r} & \text{if } k < 0. \end{cases}$$

This was first introduced in [6, Corollary 8.2] as a result of explicit computations on dimension 3 contact manifolds. We will see that it is an approprariate sub-Riemannian generalization of F_{Riem}.

In theorem we will denote succinctly the assumptions on the H-type foliation $(\mathbb{M}, \mathcal{H}, g)$ by writing (J^2) if it satisfies the J^2 condition and $(phCs)$ for the existence of a parallel horizontal Clifford structure with constant κ.

Theorem 5.3.5 ([25, Theorems 3.3, 3.4, 3.6, and 3.8])**.** *Let* $(\mathbb{M},\mathcal{H},g)$ *be an H-type foliation with horizontally parallel torsion. Fix* $x\in\mathbb{M}$ *and* $y\notin \mathbf{Cut}_\varepsilon(x)$*, and let* $\gamma\colon [0,r_\varepsilon]\to\mathbb{M}$ *be the unique unit speed* g_ε*-geodesic connecting* x *to* $y=\gamma(r_\varepsilon)$*. Then at* y*,*

(a) *Geodesic Comparison: Assume (phCs). For any unit* $X\in\operatorname{span}(\operatorname{pr}_{\mathcal{H}}\dot\gamma)$*,*

$$\operatorname{Hess}^{\nabla}(r_\varepsilon)(X,X)\le \frac{1-h^2}{r_\varepsilon}.$$

(b) *Riemannian Comparison: Assume* $(J^2, phCs)$*. Suppose there exists* $\rho\in\mathbb{R}$ *such that whenever* $Y\in\mathcal{H}, X\in\mathcal{H}_{\mathrm{Riem}}(Y)$ *it holds that* $\operatorname{Sec}(X\wedge Y)\ge\rho$*. Then for any unit* $X\in\mathcal{H}_{\mathrm{Riem}}(\dot\gamma)$

$$\operatorname{Hess}^{\nabla}(r_\varepsilon)(X,X)\le F_{\mathrm{Riem}}(r_\varepsilon,\rho h^2+\frac{1}{4}v^2).$$

(c) *Sasakian Comparison: Suppose there exists* $\rho\in\mathbb{R}$ *such that for any* $X\in\mathcal{H}, Z\in\mathcal{V}$ *it holds that* $\operatorname{Sec}(X\wedge J_Z X)\ge\rho$*. For any unit* $X\in\mathcal{H}_{\mathrm{Sas}}(\dot\gamma)$ *we can write* $X=J_Z\dot\gamma$*, and we have that*

   (i) *If* $Z\propto\operatorname{pr}_{\mathcal{V}}\dot\gamma$*,*

$$\operatorname{Hess}^{\nabla}(r_\varepsilon)(X,X)\le F_{\mathrm{Sas}}(r_\varepsilon,\rho h^2+v^2)$$

   (ii) *Assume* $(J^2, phCs)$*. If* $Z\perp\operatorname{pr}_{\mathcal{V}}\dot\gamma$*,*

$$\operatorname{Hess}^{\nabla}(r_\varepsilon)(X,X)\le F_{\mathrm{Sas}}(r_\varepsilon,\rho h^2+(2-\kappa\varepsilon)(\kappa\varepsilon-1)v^2)$$

Remark 5.3.6. Notice that that case (i) of the Sasakian comparison is singular in only requiring horizontally parallel torsion. When $m = 1$ this is particularly powerful, since it will contain all of $\mathcal{H}_{\text{Sas}}(\dot\gamma)$ (notice that if $\dot\gamma \in \mathcal{H}$ the condition $Z \propto \operatorname{pr}_{\mathcal{V}} \dot\gamma$ is trivial).

Remark 5.3.7. Notice that in the degenerate case $m = \operatorname{rank}(\mathcal{V}) = 0$ that $\mathcal{H}_{\text{Riem}}(\dot\gamma) \cong \mathcal{H} \setminus \operatorname{span}(\dot\gamma) \cong T\mathbb{M} \setminus \operatorname{span}(\dot\gamma)$ and $h = 1, v = 0$, thus the Riemannian comparison recovers the classical theorem 5.1.4. The essential conclusion is that the Sasakian directions $\mathcal{H}_{\text{Sas}}$ are those for which the sub-Riemannian structure weakens the Hessian comparison (by roughly a factor of 4 as $r_\varepsilon \to 0^+$, per remark 5.3.9).

Proof. The strategy in each case has a similar structure. We seek to apply theorem 5.2.4.

- Along the geodesic direction, denoting $\operatorname{pr}_{\dot\gamma} X = \operatorname{pr}_{\mathcal{H}} X - h^2\dot\gamma$ we can choose

$$W(t) = \frac{t}{r_\varepsilon} \operatorname{pr}_{\dot\gamma} X(t)$$

  for which the Jacobi equation is verified easily. By theorem 5.2.4 we have

$$\operatorname{Hess}(r_\varepsilon)(\operatorname{pr}_{\dot\gamma} X, \operatorname{pr}_{\dot\gamma} X) \leq \frac{1 - h^2}{r_\varepsilon}.$$

  Consideration of the symmetries of the Hessian, an application of the eikonal equation $\|\dot\gamma\|_\varepsilon = 1$, and corollary 5.2.6 completes the proof.

- For the Riemannian and Sasakian comparisons, we can use lemma 5.2.8 to simplify the Jacobi equation. If we assume that the sectional curvature is constant and choose

$$W = fX + gY$$

for an appropriate vector field Y, we can explicitly write the Jacobi equation as an ODE in f, g and pick initial conditions that will force W to satisfy theorem 5.2.4.

For example, in the Riemannian case $X \in \mathcal{H}_{\mathrm{Riem}}(\dot{\gamma})$ we choose $Y = \frac{1}{\varepsilon v} J_{\dot{\gamma}} X$ and the Jacobi equation becomes

$$\ddot{f} + v\dot{g} + \rho h^2 f = 0$$
$$\ddot{g} - v\dot{f} + \rho h^2 g = 0$$

with initial conditions $f(0) = f(r_\varepsilon(y)) - 1 = g(0) = g(r_\varepsilon(y)) = 0$.

In all cases the solution can be found explicitly, and applying theorem 5.2.4 and corollary 5.2.6 the theorem follows.

Note that in the Sasakian case $X = J_Z \dot{\gamma}$ we must distinguish between $Z \propto \mathrm{pr}_{\mathcal{V}} \dot{\gamma}$ and $Z \perp \mathrm{pr}_{\mathcal{V}} \dot{\gamma}$, with the second case being significantly more difficult. We arrive at equivalent ODEs for both; the solution of the system is a function of $\varepsilon > 0$ but can be bounded above uniformly to complete the proof.

□

Remark 5.3.8. One can also determine vertical Hessian comparison theorems using the same proof strategy outlined above. While the proof by this method splits into the cases $Z \perp \mathrm{pr}_{\mathcal{V}} \dot\gamma$ and $Z \propto \mathrm{pr}_{\mathcal{V}} \dot\gamma$, we find that in all vertical directions $Z \in \mathcal{V}$,

$$\mathrm{Hess}(r_\varepsilon)(Z, Z) \leq \frac{12}{r_\varepsilon^3}$$

which agrees with the result [23, Remark 3.10].

Remark 5.3.9. Direct computation shows that for any fixed k, we have the asymptotic relation

$$\frac{F_{\mathrm{Sas}}(r,k)}{F_{\mathrm{Riem}}(r,k)} \searrow 4 \text{ as } r \to 0^+$$

with faster convergence as $k \to 0$. Since all the curvature terms have the form $k = \rho h^2 + \alpha v^2 \xrightarrow{\varepsilon \to 0^+} \rho + \alpha \|\nabla_{\mathcal{V}} r_0\|^2$, this tells us roughly that in the sub-Riemannian limit the Hessian of the distance function grows 4 times as fast in Sasakian directions than in Riemannian directions.

In fact, this coefficient 4 is related to the measure contraction property MCP and geodesic dimension discussed in [103, 15]. The MCP is a different generalization of Ricci lower curvature bounds, and the associated geodesic dimension that arises can be thought of as a measure of the growth of geodesics. It is established in [15, Theorem 3] that H-type Carnot groups satisfy the MCP with geodesic dimension $n + 3m$; this is compatible with the above observation. See [4] for more on geodesic dimension.

5.3.3 Uniform comparison theorems

In this section we conclude our previous analysis by arriving at purely sub-Riemannian comparison theorems. The key idea here is that the results of theorem 5.3.5 are uniform in $\varepsilon > 0$, and thus they carry over in the limit $\varepsilon \to 0^+$ to the sub-Riemannian structure.

We acheive Bonnet-Myers diameter bounds comparable to those acheived in [3] on contact manifolds, [12] on quaternion contact manifolds, and [104] on 3-Sasakian manifolds using the Hamiltonian approach developed in [14]. See also [62], in which Grong develops a generalization to higher-step sub-Riemannian manifolds that agrees with the results here.

We also have a sub-Laplacian comparison that naturally arises from our Hessian bounds in an analogous way to the Riemannian Laplacian comparison theorem 5.1.4. These agree with the results of [23].

Sub-Riemannian Bonnet-Myers theorems

We begin with a simple lemma following from theorem 5.3.5.

Lemma 5.3.10. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with horizontally parallel torsion.*

(a) *Riemannian Estimate: Assume* $(J^2, phCs)$*. Suppose there exists* $\rho \in \mathbb{R}$ *such that whenever* $Y \in \mathcal{H}, X \in \mathcal{H}_{\mathrm{Riem}}(Y)$ *it holds that* $\mathrm{Sec}(X \wedge Y) \geq \rho$*, and* $K_{Riem} := \rho h^2 + \frac{1}{4}v^2 > 0$*. Then*

$$r_\varepsilon < \frac{\pi}{\sqrt{K_{Riem}}}$$

(b) *Sasakian Estimate: Suppose there exists* $\rho \in \mathbb{R}$ *such that for any* $X \in \mathcal{H}, Z \in \mathcal{V}$ *it holds that* $\mathrm{Sec}(X \wedge J_Z X) \geq \rho$ *and* $K_{Sas} := \rho h^2 + v^2 > 0$*. Then*

$$r_\varepsilon < \frac{2\pi}{\sqrt{K_{Sas}}}$$

Proof. These follow from the fact that $F_{\text{Riem}}(r,k)$ diverges for $r \geq \frac{\pi}{\sqrt{k}}$ and that $F_{\text{Sas}}(r,k)$ diverges for $r \geq \frac{2\pi}{\sqrt{k}}$. □

Denote by $\operatorname{diam}_0(\mathbb{M})$ the sub-Riemannian diameter of $\mathbb{M}$; that is,

$$\operatorname{diam}_0(\mathbb{M}) = \sup_{x,y\in\mathbb{M}} d_0(x,y).$$

We can proceed directly from the lemma to a diameter estimate for $\mathbb{M}$.

Theorem 5.3.11 ([25, Theorem 3.10(b)]). *Let $(\mathbb{M},\mathcal{H},g)$ be an H-type foliation that is complete and has horizontally parallel torsion. Assume there is some $\rho > 0$ such that for any unit $X \in \mathcal{H}, Z \in \mathcal{V}$ we have*

$$\operatorname{Sec}(X \wedge J_Z X) \geq \rho.$$

Then

$$\operatorname{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\rho}}$$

Proof. We pass to the universal cover $\tilde{\mathbb{M}}$. For $\rho > 0$, we have $K_{Sas} = \rho h^2 + v^2 > 0$ and from case (b) of lemma 5.3.10 we have uniform convergence

$$\frac{2\pi}{\sqrt{K_{Sas}}} \xrightarrow{\varepsilon\to 0^+} \frac{2\pi}{\sqrt{\rho + \|\nabla_{\mathcal{V}} r_0\|^2}} \leq \frac{2\pi}{\sqrt{\rho}}$$

completing the proof. □

We can be more delicate, considering instead a condition on the horizontal Ricci curvature

$$\operatorname{Ric}_{\mathcal{H}}(X,Y) = \sum_{i=1}^{n} g_{\mathcal{H}}(R^{\nabla}(W_i,X)Y,W_i)$$

where the W_i form a g-orthonormal basis of $\mathcal{H}$.

We decompose $\mathrm{Ric}_{\mathcal{H}}(X,X)$ along the canonical decomposition for $X \in \mathcal{H}$ as $\mathrm{Ric}_{\mathrm{Sas}}(X,X) + \mathrm{Ric}_{\mathrm{Riem}}(X,X)$ (observe that $g_{\mathcal{H}}(R^{\nabla}(X,X)X,X) = 0$ and so the span(X) term vanishes). More precisely,

$$\mathrm{Ric}_{Sas}(X,X) = \sum_{i=1}^{m} g_{\mathcal{H}}(R^{\nabla}(J_{Z_i}X,X)X, J_{Z_i}X)$$
$$\mathrm{Ric}_{Riem}(X,X) = \sum_{i=1}^{n-m-1} g_{\mathcal{H}}(R^{\nabla}(Y_i,X)X,Y_i)$$

where the Z_i form a $g_{\mathcal{V}}$-orthonormal basis for $\mathcal{V}$ and the Y_i form a $g_{\mathcal{H}}$-orthonormal basis for $\mathcal{H}_{\mathrm{Riem}}(X)$.

Theorem 5.3.12 ([25, Theorem 3.10(a,c)])**.** *Let* $(\mathbb{M},\mathcal{H},g)$ *be an H-type foliation that is complete, has horizontally parallel Clifford structure, and satisfies the* J^2 *condition.*

(a) *Riemannian-type Diameter Estimate: Suppose that* $n > m+1$ *and for any* $X \in \mathcal{H}$ *there is a* $\rho > 0$ *such that*

$$\mathrm{Ric}_{Riem}(X,X) \geq (n-m-1)\rho\|X\|^2.$$

Then

$$\mathrm{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho}}$$

(b) *Sasakian-type Diameter Estimate: Suppose that for any* $X \in \mathcal{H}$ *there is a* $\rho > 0$ *such that*

$$\mathrm{Ric}_{Sas}(X,X) \geq m\rho\|X\|^2.$$

and either

(i) *For any* $X \in \mathcal{H}$ *and unit* $Z \in \mathcal{V}$, $\mathrm{Sec}(X \wedge J_Z X) \geq 0$, *OR*

(ii) $n > m + 1$ *and for any* $X \in \mathcal{H}$, $\mathrm{Ric}_{Riem}(X, X) \geq 0$.

Then

$$\mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi\sqrt{3}}{\sqrt{\rho}}$$

Proof. Part (a) proceeds as in theorem 5.3.11, applying lemma 5.3.10 (a) after the observation that the assumption

$$\mathrm{Ric}_{Riem}(X, X) \geq (n - m - 1)\rho\|X\|^2$$

together with the J^2 condition implies the existence of an orthonormal basis X_i for $\mathcal{H}_{\mathrm{Riem}}(X)$ such that

$$\sum_{i=1}^{n-m-1} \mathrm{Sec}(X \wedge X_i) \geq (n - m - 1)\rho.$$

Part (b) is more subtle, we refer to [25]. □

It's interesting to note that a lower bound on the Riemannian Ricci curvature gives a sharp diameter estimate on the complex Hopf fibration, but a lower bound on the Sasakian Ricci curvature does not. It remains to be seen if the method can be improved. The diameter bound from theorem 5.3.11 is sharp on quaternionic and octonionic Hopf fibrations, but undesirably requires a condition on the sectional curvature. See [27, 29, 19].

Remark 5.3.13. Observing that $\mathrm{Ric}_{\mathcal{H}}$ is the same curvature as in theorem 3.4.7, the

Horizontal Einstein condition implies that

$$\mathrm{Ric}_{Sas}(X,X)+\mathrm{Ric}_{Riem}(X,X)=\lambda\|X\|^2.$$

This is insufficient to imply theorem 5.3.12 by itself, but if we assume $n>m+1$ and the Horizontal Einstein condition holds with $\lambda>0$ then

- $\mathrm{Ric}_{Sas}(X,X)\geq 0 \implies \mathrm{diam}_0(\mathbb{M})\leq \pi\sqrt{\frac{m}{\lambda}}$
- $\mathrm{Ric}_{Riem}(X,X)\geq 0 \implies \mathrm{diam}_0(\mathbb{M})\leq 2\pi\sqrt{3}\sqrt{\frac{n-m-1}{\lambda}}$.

Because the proofs of theorem 5.3.11 and theorem 5.3.12 pass from $\mathbb{M}$ to the universal cover $\tilde{\mathbb{M}}$, we also have that

Corollary 5.3.14. *If theorem 5.3.11, theorem 5.3.12(a), or theorem 5.3.12(b) hold, then* $(\mathbb{M},\mathcal{H},g)$ *is compact with finite fundamental group.*

Sub-Laplacian comparison theorems

Recall from section 3.2.3 the horizontal Laplacian $\Delta_{\mathcal{H}}$ defined as the horizontal trace of the Hessian

$$\Delta_{\mathcal{H}}u=\sum_{i=1}^{n}\mathrm{Hess}^{\nabla}(u)(X_i,X_i)$$

for a g-orthonormal basis X_i of $\mathcal{H}$. We can decompose the Hessian using the canonical decomposition for any $Y\in\mathcal{H}$ and thereby obtain sub-Laplacian comparison theorems for $\Delta_{\mathcal{H}}r_\varepsilon$. That is, we observe that

$$\Delta_{\mathcal{H}}r_\varepsilon=\mathrm{Hess}^{\nabla}(r_\varepsilon)(\nabla_{\mathcal{H}}r_\varepsilon,\nabla_{\mathcal{H}}r_\varepsilon)+\sum_{i=1}^{m}\mathrm{Hess}^{\nabla}(r_\varepsilon)(J_{Z_i}\nabla_{\mathcal{H}}r_\varepsilon,J_{Z_i}\nabla_{\mathcal{H}}r_\varepsilon)+\sum_{i=1}^{n-m-1}\mathrm{Hess}^{\nabla}(r_\varepsilon)(X_i,X_i)$$

where the Z_i form a g-orthogonal basis for $\mathcal{V}$ with $\|Z_i\| = \frac{1}{\|\nabla_{\mathcal{H}} r_\varepsilon\|}$ and the X_i form a g-orthonormal basis for $\mathcal{H}_{\text{Riem}}(\nabla_{\mathcal{H}} r_\varepsilon)$.

Combining the above results, we can establish a sub-Laplacian comparison theorem as $\varepsilon \to 0^+$.

Theorem 5.3.15 ([25, Theorem 3.12])**.** *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure satisfying the J^2 condition. Let $x \in \mathbb{M}$ and define $r_0(y) = d_0(x, y)$. Assume there exists $\rho > 0$ such that*

$$\text{Sec}(X \wedge Y) \geq \rho$$

for all $X, Y \in \mathcal{H}$. For $y \notin \mathbf{Cut}_0(x)$ we have

$$\Delta_{\mathcal{H}} r_0 \leq (n - m - 1)F_{\text{Riem}}(r_0, K_{\text{Riem}}) + F_{\text{Sas}}(r_0, K_{\text{Sas},\dot{\gamma}}) + (m - 1)F_{\text{Sas}}(r_0, K_{\text{Sas},\perp})$$

where

$$K_{\text{Riem}} = \rho + \frac{1}{4}\|\nabla_{\mathcal{V}} r_0\|^2$$
$$K_{\text{Sas},\dot{\gamma}} = \rho + \|\nabla_{\mathcal{V}} r_0\|^2$$
$$K_{\text{Riem},\perp} = \rho - 2\|\nabla_{\mathcal{V}} r_0\|^2$$

The highlight of the theorem being that it is completely independent of the metric structure on $\mathcal{V}$, and so it is a purely sub-Riemannian result.

Proof. The proof has two key steps. First one achieves an analogous comparison theorem for $\Delta_{\mathcal{H}} r_\varepsilon$ in the domain $\varepsilon > 0$ simply by summing the results of theorem 5.3.5. It holds the for $y \notin \mathbf{Cut}_\varepsilon(y)$ that we have uniform convergence $r_\varepsilon \to r_0$ (this is highly nontrivial, [25, Lemma A.1] for details) and as a consequence we have convergence

$$\lim_{\varepsilon\to 0^+} \nabla_{\mathcal{H}} r_\varepsilon = \nabla_{\mathcal{H}} r_0 \qquad \lim_{\varepsilon\to 0^+} \nabla_{\mathcal{V}} r_\varepsilon = \nabla_{\mathcal{V}} r_0.$$

We also have that the eikonal equation $\|\nabla_{\mathcal{H}} r_\varepsilon\|^2 + \varepsilon\|\nabla_{\mathcal{V}} r_\varepsilon\|^2 = 1$ for $\varepsilon > 0$ implies that $\|\nabla_{\mathcal{H}} r_0\| = 1$. Applying this with the comparison for $\Delta_{\mathcal{H}} r_\varepsilon$ gives the theorem, taking $\varepsilon \to 0^+$. $\square$

Chapter 6

Future Research Directions

In this chapter we briefly consider possible directions for furthering this work.

Berger-Simons Holonomy Theorem

Considering the bijection between the horizontal holonomy of H-type submersions and the Riemannian holonomy of their base spaces achieved in theorem 4.3.7, it is possible that we might recover a partial proof of the Berger-Simons theorem 4.1.6. It's clear from the classification result theorem 3.2.12 that we cannot hope to find the exceptional holonomy G_2 via this approach since there is no H-type submersion with this horizontal holonomy. However, improving theorem 4.3.11 could allow for a recovery of the rest of the Berger-Simons classification. If successful, this would provide a complementary geometric proof to that of Olmos [93], which was accomplished on submanifolds.

Index results for the sub-Laplacian

The index theory of sub-Riemannian geometry is a very open field. There are many obstructions, such as the fact that in the famous Atiyah-Singer theorem

$$\mathrm{ind}(D) = \int_{\mathbb{M}} \mathrm{ch}(D)\,\mathrm{Td}(\mathbb{M})$$

it's not possible to define the Chern class $\mathrm{ch}(D)$ for a hypoelliptic operator such as the sub-Laplacian. In particular, heat kernel approaches to the index theorem require a decomposition of the Laplacian as the square of a Dirac operator, as

$$\Delta = (d + \delta)^2$$

where δ is the formal adjoint of the exterior derivative d by the Riemannian metric; this isn't sensible for a sub-Riemannian metric, as it is necessarily singular.

However, in the context of H-type foliations one can consider δ_ε, defined for $\varepsilon > 0$ as the formal adjoint of d by the Riemannian metric g_ε. It seems possible to recover uniform results on the behavior of these objects as was the approach for the comparison theorems theorem 5.3.12 and theorem 5.3.15; if so, one could hope to thereby achieve a McKean-Singer supertrace theorem in the $\varepsilon \to 0^+$ limit, leading potentially to an index theory for the sub-Laplacian.

Higher step sub-Riemannian manifolds

H-type foliations are necessarily models of 2-step sub-Riemannian structures because of their definition using the Bott connection. By defining an analogous construction using the Hladky connection 2.2.33 on higher-step sub-Riemannian manifolds with an appropriate complementary Riemannian structure (which is not necessarily a foliation), it is possible that we could study the action of a graded family of Clifford algebras on the horizontal distribution and thereby achieve results for higher-step sub-Riemannian manifolds. It is presently unclear how rigid this construction would be.

Bibliography

[1] A. Agrachev. Any sub-Riemannian metric has points of smoothness. *Rossiiskaya Akademiya Nauk. Doklady Akademii Nauk*, 424(3):295–298, Feb. 2009.

[2] A. Agrachev, D. Barilari, and U. Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Cambridge University Press, 2019.

[3] A. Agrachev, D. Barilari, and L. Rizzi. Sub-Riemannian curvature in contact geometry. *Journal of Geometric Analysis*, 27(1):366–408, 2017.

[4] A. Agrachev, D. Barilari, and L. Rizzi. Curvature: a variational approach. *Memoirs of the American Mathematical Society*, 256(1225):v+142, 2018.

[5] A. Agrachev and P. W. Y. Lee. Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds. *Mathematische Annalen*, 360(1-2):209–253, 2014.

[6] A. Agrachev and P. W. Y. Lee. Bishop and Laplacian comparison theorems on three-dimensional contact sub-Riemannian manifolds with symmetry. *Journal of Geometric Analysis*, 25(1):512–535, 2015.

[7] A. Agrachev and Y. L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[8] W. Ambrose and I. M. Singer. A theorem on holonomy. *Transactions of the American Mathematical Society*, 75(3):428–428, Mar. 1953.

[9] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lecture Notes in Mathematics*, pages 1–114. Springer Berlin Heidelberg, 1994.

[10] D. Bakry and M. Ledoux. Sobolev inequalities and myers's diameter theorem for an abstract markov generator. *Duke Mathematical Journal*, 85(1):253–270, Oct. 1996.

[11] A. Banyaga and D. F. Houenou. *A Brief Introduction to Symplectic and Contact Manifolds*, volume 15 of *Nankai Tracts in Mathematics*. World Scientific, Aug. 2016.

[12] D. Barilari and S. Ivanov. A Bonnet-Myers type theorem for quaternionic contact structures. *Calculus of Variations and Partial Differential Equations*, 58(1), Jan. 2019.

[13] D. Barilari and L. Rizzi. A formula for Popp's volume in sub-Riemannian geometry. *Analysis and Geometry in Metric Spaces*, 1:42–57, 2013.

[14] D. Barilari and L. Rizzi. Comparison theorems for conjugate points in sub-Riemannian geometry. *ESAIM. Control, Optimisation and Calculus of Variations*, 22(2):439–472, 2016.

[15] D. Barilari and L. Rizzi. Sharp measure contraction property for generalized H-type Carnot groups. *Communications in Contemporary Mathematics*, 20(06):1750081, Aug. 2018.

[16] D. Barilari and L. Rizzi. Bakry-émery curvature and model spaces in sub-riemannian geometry. *Mathematische Annalen*, 377(1-2):435–482, mar 2020.

[17] F. Baudoin. Sub-Laplacians and hypoelliptic operators on totally geodesic Riemannian foliations. In *Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. 1*, EMS Ser. Lect. Math., pages 259–321. Eur. Math. Soc., Zürich, 2016.

[18] F. Baudoin, M. Bonnefont, and N. Garofalo. A sub-Riemannian curvature-dimension inequality, volume doubling property and the Poincaré inequality. *Mathematische Annalen*, 358(3-4):833–860, Sept. 2014.

[19] F. Baudoin and G. Cho. The subelliptic heat kernel of the octonionic Hopf fibration. *arXiv e-prints*, Apr. 2019.

[20] F. Baudoin, N. Demni, and J. Wang. The horizontal heat kernel on the quaternionic anti de-sitter spaces and related twistor spaces. *arXiv preprint*, 2018.

[21] F. Baudoin and N. Garofalo. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. *Journal of the European Mathematical Society (JEMS)*, 19(1):151–219, 2017.

[22] F. Baudoin and E. Grong. Transverse Weitzenböck formulas and de Rham cohomology of totally geodesic foliations. *ArXiv e-prints*, Nov. 2017.

[23] F. Baudoin, E. Grong, K. Kuwada, and A. Thalmaier. Sub-Laplacian comparison theorems on totally geodesic Riemannian foliations. *Calculus of Variations and Partial Differential Equations*, 58(4), 2019.

[24] F. Baudoin, E. Grong, G. Vega-Molino, and L. Rizzi. H-type foliations. *arXiv e-prints*, Dec. 2018.

[25] F. Baudoin, E. Grong, G. Vega-Molino, and L. Rizzi. Comparison theorems on h-type sub-riemannian manifolds. *arXiv e-prints*, Sept. 2019.

[26] F. Baudoin, B. Kim, and J. Wang. Transverse Weitzenböck formulas and curvature dimension inequalities on Riemannian foliations with totally geodesic leaves. *Communications in Analysis and Geometry*, 24(5):913–937, 2016.

[27] F. Baudoin and J. Wang. The subelliptic heat kernel on the CR sphere. *Mathematische Zeitschrift*, 275(1-2):135–150, 2013.

[28] F. Baudoin and J. Wang. Curvature dimension inequalities and subelliptic heat kernel gradient bounds on contact manifolds. *Potential Analysis. An International Journal Devoted to the Interactions between Potential Theory, Probability Theory, Geometry and Functional Analysis*, 40(2):163–193, 2014.

[29] F. Baudoin and J. Wang. The subelliptic heat kernels of the quaternionic Hopf fibration. *Potential Analysis*, 41(3):959–982, May 2014.

[30] A. Bejancu and H. R. Farran. *Foliations and Geometric Structures*. Springer Netherlands, Nov. 2005.

[31] A. Bejancu and H. R. Farran. On totally geodesic foliations with bundle-like metric. *Journal of Geometry*, 85(1-2):7–14, Sept. 2006.

[32] A. Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.

[33] M. Berger. Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bulletin de la Societe Mathematique de France*, 83:279–330, 1955.

[34] M. Berger. An extension of rauch's metric comparison theorem and some applications. *Illinois Journal of Mathematics*, 6(4):700–712, dec 1962.

[35] A. L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[36] O. Biquard. *Metriques D'Einstein Asymptotiquement Symetriques*, volume 265 of *Asterisque*. Societe Mathematique de France, 2000.

[37] S. Bochner. Vector fields and Ricci curvature. *Bull. Amer. Math. Soc.*, 52:776–797, 1946.

[38] N. Bourbaki. *Éléments de mathématique. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 9: Formes sesquilinéaires et formes quadratiques.* Springer-Verlag GmbH, 1959.

[39] C. P. Boyer and K. Galicki. 3-Sasakian manifolds. In *Surveys in differential geometry: essays on Einstein manifolds*, volume 6 of *Surv. Differ. Geom.*, pages 123–184. Int. Press, Boston, MA, 1999.

[40] C. P. Boyer and K. Galicki. *Sasakian geometry*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[41] R. L. Bryant. Metrics with exceptional holonomy. *The Annals of Mathematics*, 126(3):525, Nov. 1987.

[42] R. L. Bryant and S. M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Mathematical Journal*, 58(3):829–850, June 1989.

[43] G. Bădiţoiu and S. Ianuş. Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces. *Differential Geometry and its Applications*, 16(1):79–94, 2002.

[44] E. Calabi. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *The Michigan Mathematical Journal*, 5(2):105–126, 1958.

[45] O. Calin, D.-C. Chang, and I. Markina. Geometric analysis on H-type groups related to division algebras. *Mathematische Nachrichten*, 282(1):44–68, 2009.

[46] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, May 2007.

[47] É. Cartan. Sur une classe remarquable d'espaces de riemann. *Bulletin de la Société Mathématique de France*, 2:214–264, 1926.

[48] É. Cartan. Sur une classe remarquable d'espaces de riemann. II. *Bulletin de la Société Mathématique de France*, 2:114–134, 1927.

[49] D.-C. Chang, I. Markina, and A. Vasil'ev. Sub-lorentzian geometry on anti-de sitter space. *Journal de Mathématiques Pures et Appliquées*, 90(1):82–110, July 2008.

[50] S.-Y. Cheng. Eigenvalue comparison theorems and its geometric applications. *Mathematische Zeitschrift*, 143(3):289–297, oct 1975.

[51] W.-L. Chow. Über systeme von liearren partiellen differentialgleichungen erster ordnung. *Mathematische Annalen*, 117-117(1):98–105, dec 1940.

[52] M. Cowling, A. H. Dooley, A. Korányi, and F. Ricci. H-type groups and Iwasawa decompositions. *Advances in Mathematics*, 87(1):1–41, 1991.

[53] G. deRham. Sur la réductibilité d'un espace de riemann. *Commentarii Mathematici Helvetici*, 26(1):328–344, Dec. 1952.

[54] S. Deshmukh and V. A. Khan. Geodesic vector fields and eikonal equation on a riemannian manifold. *Indagationes Mathematicae*, 30(4):542–552, July 2019.

[55] S. Dragomir and G. Tomassini. *Differential geometry and analysis on CR manifolds*, volume 246 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, Mar. 2006.

[56] D. Duchemin. Quaternionic contact structures in dimension 7. *Annales de l'institut Fourier*, 56(4):851–885, 2006.

[57] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[58] G. Frobenius. Ueber das pfaffsche problem. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1877(82):230–315, Jan. 1877.

[59] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian Geometry*. Springer-Verlag, Berlin Heidelberg, 2004.

[60] A. Gray. Spaces of constancy of curvature operators. *Proceedings of the American Mathematical Society*, 17:897–902, 1966.

[61] E. Grong. Model spaces in sub-Riemannian geometry. *arXiv e-prints*, 1610.07359, 2016. To appear in: Communications in Analysis and Geometry.

[62] E. Grong. Affine connections and curvature in sub-riemannian geometry. *arXiv e-prints*, Jan. 2020.

[63] E. Grong and A. Thalmaier. Curvature-dimension inequalities on sub-Riemannian manifolds obtained from Riemannian foliations: part I. *Mathematische Zeitschrift*, 282(1-2):99–130, sep 2015.

[64] E. Grong and A. Thalmaier. Curvature-dimension inequalities on sub-Riemannian manifolds obtained from Riemannian foliations: part II. *Mathematische Zeitschrift*, 282(1-2):131–164, sep 2015.

[65] C. Hadfield. Twistor spaces over quaternionic-kähler manifolds. Master's thesis, University Pierre & Marie Curie, 2014.

[66] R. Hermann. On the differential geometry of foliations. *The Annals of Mathematics*, 72(3):445, Nov. 1960.

[67] G. Hernandez. On hyper f-structures. *Mathematische Annalen*, 306(2):205–230, 1996.

[68] R. K. Hladky. Connections and curvature in sub-Riemannian geometry. *Houston Journal of Mathematics*, 38(4):1107–1134, Dec. 2012.

[69] L. Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119:147–171, 1967.

[70] W. K. Hughen. *The sub-Riemannian geometry of three-manifolds*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Duke University.

[71] S. Ishihara. Quaternion Kählerian manifolds. *J. Differential Geometry*, 9:483–500, 1974.

[72] W. Jelonek. Positive and negative 3-K-contact structures. *Proceedings of the American Mathematical Society*, 129(1):247–256, 2001.

[73] D. L. Johnson and L. B. Whitt. Totally geodesic foliations. *Journal of Differential Geometry*, 15(2):225–235, 1980.

[74] A. Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Transactions of the American Mathematical Society*, 258(1):147–153, 1980.

[75] A. Kaplan. Riemannian nilmanifolds attached to clifford modules. *Geometriae Dedicata*, 11(2), June 1981.

[76] A. Kaplan. On the geometry of groups of heisenberg type. *Bulletin of the London Mathematical Society*, 15(1):35–42, Jan. 1983.

[77] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, Feb. 1996.

[78] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. II.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996.

[79] A. Korányi. Geometric properties of heisenberg-type groups. *Advances in Mathematics*, 56(1):28–38, Apr. 1985.

[80] S. Lang. *Introduction to Differentiable Manifolds.* Universitext. Springer New York, second edition, Oct. 2002.

[81] H. B. Lawson and M.-L. Michelsohn. *Spin Geometry.* Princeton University Press, Feb. 1990.

[82] M. Ledoux. The geometry of markov diffusion generators. *Annales de la faculté des sciences de Toulouse Mathématiques*, 9(2):305–366, 2000.

[83] J. M. Lee. *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1997.

[84] P. W. Y. Lee. Ricci curvature lower bounds on sasakian manifolds. *arXiv e-prints*, Nov. 2015.

[85] P. W. Y. Lee, C. Li, and I. Zelenko. Ricci curvature type lower bounds for sub-Riemannian structures on Sasakian manifolds. *Discrete and Continuous Dynamical Systems. Series A*, 36(1):303–321, 2016.

[86] I. Moerdijk and J. Mrcun. *Introduction to Foliations and Lie Groupoids.* Cambridge University Press, Apr. 2007.

[87] P. Molino. *Riemannian foliations*, volume 73 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1988.

[88] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, R.I, 2002.

[89] A. Moroianu and U. Semmelmann. Clifford structures on riemannian manifolds. *Advances in Mathematics*, 228(2):940–967, Oct. 2011.

[90] M. I. Munteanu. *One-dimensional Riemannian foliations on the Heisenberg group*. phdthesis, University of Oklahoma, 2002.

[91] S. B. Myers. Riemannian manifolds with positive mean curvature. *Duke Mathematical Journal*, 8(2):401–404, June 1941.

[92] M. Nagase and D. Sasaki. Hermitian tanno connection and bochner type curvature tensors of contact riemannian manifolds. *The University of Tokyo. Journal of Mathematical Sciences*, 25(2):149–169, 2018.

[93] C. Olmos. A geometric proof of the berger holonomy theorem. *Annals of Mathematics*, 161(1):579–588, Jan. 2005.

[94] L. Ornea, M. Parton, P. Piccinni, and V. Vuletescu. Spin(9) geometry of the octonionic Hopf fibration. *Transformation Groups*, 18(3):845–864, 2013.

[95] A. M. Pastore, M. Falcitelli, and S. Ianuş. *Riemannian Submersions and Related Topics*. World Scientific Publishing Co. Inc., June 2004.

[96] P. Petersen. *Riemannian Geometry*. Springer-Verlag GmbH, Mar. 2016.

[97] D. Prandi, L. Rizzi, and M. Seri. A sub-riemannian santaló formula with applications to isoperimetric inequalities and first dirichlet eigenvalue of hypoelliptic operators. *Journal of Differential Geometry*, 111(2):339–379, Feb. 2019.

[98] P. K. Rashevsky. Any two points of a totally nonholonomic space may be connected by an admissible line. *Uch. Zap. Ped. Inst. im. Liebknechta, Ser. Phys. Math*, 2:83–94, 1938. (in Russian).

[99] H. E. Rauch. A contribution to differential geometry in the large. *The Annals of Mathematics*, 54(1):38, July 1951.

[100] L. Rifford. Ricci curvatures in Carnot groups. *Mathematical Control and Related Fields*, 3(4):467–487, 2013.

[101] L. Rifford. *Sub-Riemannian geometry and optimal transport*. SpringerBriefs in Mathematics. Springer, Cham, Apr. 2014.

[102] L. Rifford and E. Trélat. Morse-Sard type results in sub-Riemannian geometry. *Mathematische Annalen*, 332(1):145–159, 2005.

[103] L. Rizzi. Measure contraction properties of Carnot groups. *Calculus of Variations and Partial Differential Equations*, 55(3), May 2016.

[104] L. Rizzi and P. Silveira. Sub-riemannian ricci curvatures and universal diameter bounds for 3-sasakian manifolds. *Journal of the Institute of Mathematics of Jussieu*, 18(4):783–827, June 2017.

[105] S. Salamon. Quaternionic Kähler manifolds. *Inventiones Mathematicae*, 67(1):143–171, 1982.

[106] J. Simons. On the transitivity of holonomy systems. *The Annals of Mathematics*, 76(2):213, Sept. 1962.

[107] N. Tanaka. On non-degenerate real hypersurfaces, graded lie algebras and cartan connections. *Japanese journal of mathematics. New series*, 2(1):131–190, 1976.

[108] S. Tanno. Variational problems on contact Riemannian manifolds. *Transactions of the American Mathematical Society*, 314(1):349–379, 1989.

[109] S. Tanno. Remarks on a triple of K-contact structures. *The Tohoku Mathematical Journal. Second Series*, 48(4):519–531, 1996.

[110] P. Tondeur. *Foliations on Riemannian Manifolds.* Universitext. Springer Verlag, 1988.

[111] P. Tondeur. *Geometry of Foliations.* Birkhäuser Basel, Oct. 2012.

[112] J. Wang. The subelliptic heat kernel on the anti-de Sitter space. *Potential Analysis*, 45(4):635–653, Apr. 2016.

[113] F. Warner. *Foundations of differentiable manifolds and Lie groups.* Springer, New York, 1983.

[114] S. M. Webster. Pseudo-hermitian structures on a real hypersurface. *Journal of Differential Geometry*, 13(1):25–41, 1978.

[115] A. Weinstein. Symplectic manifolds and their lagrangian submanifolds. *Advances in Mathematics*, 6(3):329–346, June 1971.

[116] K. Yano and S. Bochner. *Curvature and Betti numbers.* Annals of Mathematics Studies, No. 32. Princeton University Press, Princeton, N. J., 1953.