A Brief Introduction to Morse Theory

Gianmarco Molino

University of Connecticut

October 27, 2017

What is Morse Theory?

In the following, let M be a closed, n-dimensional smooth manifold.

- Initiated by Marston Morse, 1920-1930.
- Study of critical points of smooth functions $f : M \to \mathbb{R}$.
- Attempts to recover topological information about M.

Definitions

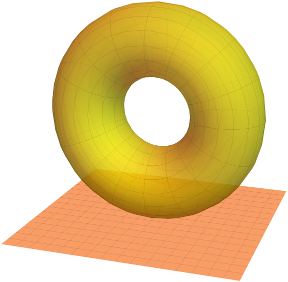
- A smooth manifold M is a topological manifold with compatible smooth atlas.
- A critical point $p \in M$ of a smooth function $f: M \to \mathbb{R}$ is a zero of the differential df.
- The Hessian $H_p(f)$ of f at a critical point $p \in M$ is the matrix of second derivatives. (Independent of coordinate system at critical points.)

Morse Functions

- A smooth function $f : M \to \mathbb{R}$ is called Morse if its critical points are isolated and nondegenerate (that is, the Hessian of f is nonsingular.)
  - Remark: Nondegenerate critical points are necessarily isolated.
- The index $\lambda(p)$ of a critical point p is the dimension of the negative eigenspace of $H_p(f)$.

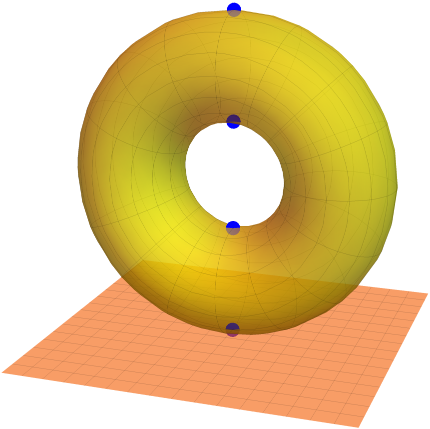
Torus with height function

Consider the 2-dimensional torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$ and a tangent plane:



Define $f : \mathbb{T}^2 \to \mathbb{R}$ to be the height above the plane.

- The function h has 4 critical points, a, b, c, d, with $\lambda(a) = 0, \lambda(b) = \lambda(c) = 1, \lambda(d) = 2$.



Morse Lemma

- Nondegeneracy of critical points is a generalization of non-vanishing of the second derivative of functions $f : \mathbb{R} \to \mathbb{R}$.
- We thus expect to be able to describe M in relation to these points.

Morse Lemma

Theorem (Lemma of Morse)

Let $f \in C^\infty(M, \mathbb{R})$, and let $p \in M$ be a nondegenerate critical point of f. Then there exists a neighborhood $U \subset M$ of p and a coordinate system $(y^1, \dots, y^n)$ on U such that $y^i(p) = 0$ for all $1 \leq i \leq n$, and moreover

$$f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$$

where $\lambda = \lambda(p)$ is the index of p.

Corollary

If $p \in M$ is a nondegenerate critical point of f, then it is isolated.

Given $f\colon M \to \mathbb{R}$, define the 'half-space'

$$M^a = f^{-1}(-\infty, a] = \{x \in M\colon f(x) \leq a\}.$$

Theorem (Milnor)

Let $f\colon M \to \mathbb{R}$ be C^∞. If $f^{-1}([a,b])$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b and furthermore M^a is a deformation retract of M^b.

The gradient of f induces a local 1-parameter family of diffeomorphisms $\phi_t : M \to M$ away from critical points. This allowing the points of M^a to flow along these gives the desired deformation retract.
Remark: The condition that $f^{-1}([a, b])$ be compact cannot be relaxed.

Theorem (Milnor)

Let $f : M \to \mathbb{R}$ be C^∞ and let $p \in M$ be a (nondegenerate, isolated) critical point of f. Set $c = f(p)$ and $\lambda = \lambda(p)$ to be the index of p. Suppose there exists $\epsilon > 0$ such that $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains no critical points of f other than p. Then for all sufficiently small ϵ, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.

The key observation is that when crossing a critical point, the Morse Lemma is applicable. It can be shown that attaching a λ-cell e^λ to $M^{c-\epsilon}$ along the $(y^1, \ldots, y^\lambda)$ axis,

$$M^{c-\epsilon} \cup e^\lambda \cong M^{c+\epsilon}.$$

Intuitively then, the manifold can be constructed from cells determined by the indices of the critical points.

Theorem (Milnor)

If $f : M \to \mathbb{R}$ is Morse and for all $a \in \mathbb{R}$ it holds that M^a is compact, then M has the homotopy type of a CW complex with one cell of dimension λ for each critical point with index λ.

This is enough to get a few results. For example,

Theorem (Reeb)

Let M be a compact smooth manifold, and let $f : M \to \mathbb{R}$ be Morse. If f has only two (nondegenerate) critical points, then M is homeomorphic to a sphere.

Differential Forms

Recall the space $\Omega^k(M)$ of differential k-forms over M, and the exterior derivative $d\colon \Omega^k \to \Omega^{k+1}$, which gives rise to the deRham co-chain complex

$$0 \to \cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots \to 0$$

Betti Numbers

The associated cohomology group is the deRham cohomology group

$$H^k_{dR}(M) = \frac{\ker d\colon \Omega^k \to \Omega^{k+1}}{\operatorname{im} d\colon \Omega^{k-1} \to \Omega^k}$$

and further we define the k-th Betti number of M,

$$\beta_k = \dim H^k_{dR}(M).$$

This cohomology encodes topological information about the manifold algebraically, and is the starting point for fields such as Hodge Theory and Index Theory.

The Betti numbers are topological invariants. They are related to the classical Euler characteristic $\chi(M)$ by

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \beta_k.$$

Which is an explicit expression for the following lemma from Index Theory:

Lemma

Let $D = d + \delta$ be the Dirac operator for the Hodge Laplacian $\Delta = D^2 = d\delta + \delta d$. Then

$$\chi(M) = \mathrm{index}(D)$$

where $\mathrm{index}(D) = \dim\ker(D) - \dim\mathrm{coker}(D)$ *denotes the analytic index.*

Unfortunately, the Betti numbers can be remarkably difficult to compute directly. This is where Morse Theory provides a solution.

Weak Morse Inequalities

Let $f : M \to \mathbb{R}$ be Morse, and define the Morse numbers, M_k, by

$$M_k = \#\{p \in M, df(p) = 0, \lambda(p) = k\}$$

Theorem (Weak Morse Inequalities)

Let M be compact, β_i be the Betti numbers of M, $f : M \to \mathbb{R}$ be Morse, and M_k be the Morse numbers of f. Then

$$\beta_k \leq M_k$$

and moreover

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \beta_k = \sum_{k=0}^{n} (-1)^k M_k.$$

Witten's Proof

We sketch the idea of Edward Witten's remarkable proof:
By a result from Hodge Theory,

$$\beta_k = \dim \ker \Delta \colon \Omega^k \to \Omega^k.$$

Let f be Morse. Then we define the 'twisted exterior derivative'

$$d_t = e^{-tf} d e^{tf}$$

from which we can construct the 'Witten Laplacian'

$$\Delta_t = d_t \delta_t + \delta_t d_t \colon \Omega^k \to \Omega^k,$$

There is an induced co-chain complex

$$0 \to \cdots \xrightarrow{d_t} \Omega^k(M) \xrightarrow{d_t} \Omega^{k+1}(M) \xrightarrow{d} \cdots \to 0$$

which is isomorphic to the deRham complex, so that

$$\beta_k = \dim \ker \Delta^k = \dim \ker \Delta_t^k.$$

But this is a remarkable improvement, leading to the conclusion that as $t \to \infty$ the elements of the kernel of Δ_t will concentrate around the critical points of f. Computations can then be approximated in local coordinates, leading to the Morse Inequalities.

Torus Example

The Weak Morse Inequalities give good estimates on the the Betti numbers. For example, we have for $\mathbb{T}^2$

$$\begin{aligned} \beta_0 &\leq M_0 = 1 \\ \beta_1 &\leq M_1 = 2 \\ \beta_2 &\leq M_2 = 1 \\ \chi(\mathbb{T}^2) &= M_0 - M_1 + M_2 = 1 - 2 + 1 = 0, \end{aligned}$$

using the height function from before.

Strong Morse Inequalities

We can make the inequalities sharper.

Theorem (Strong Morse Inequalities)

Let M be compact, β_i be the Betti numbers of M, $f : M \to \mathbb{R}$ be Morse, and M_k be the Morse numbers of f. Then for any $0 \leq k \leq n$,

$$\beta_k - \beta_{k-1} + \cdots \pm \beta_0 \leq M_k - M_{k-1} + \cdots \pm M_0$$

Polynomial Morse Inequalities

Define the Poincaré Polynomial $\mathcal{P}_t = \sum_{i=0}^n \beta_i t^i$ and the Morse Polynomial $\mathcal{M}_t = \sum_{i=0}^n M_i t^i$.

Theorem (Polynomial Morse Inequalities)

Assumptions as before. For $t \in \mathbb{R}$ there exist some non-negative integers Q_i such that

$$\mathcal{M}_t - \mathcal{P}_t = (1+t)\sum_{i=0}^{n-1} Q_i t^i$$

Lemma (Banyaga)

The Strong Morse Inequalities and the Polynomial Morse Inequalities are equivalent.

Raoul Bott writes (Morse Theory Indomitable):
"The $(1+t)$ term on the right gives this inequality much more power than it would have without it. The $(1+t)$ term feeds back information from the critical points of f to the topology of M."

Existence of Morse Functions

So, given a manifold M and a Morse function f we have nice results, but can we actually find Morse functions?

Yes. In fact, there is an 'easy' construction:

Theorem (Milnor)

Let M be a compact smooth manifold, and $\iota\colon M \to \mathbb{R}^N$ be an embedding of M into $\mathbb{R}^N$. For $p \in \mathbb{R}^N$, define $L_p\colon M \to \mathbb{R}$ by

$$L_p(q) = \|p - \iota(q)\|^2$$

where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^N$. Then L_p is Morse for almost every $p \in \mathbb{R}^N$.

Corollary

On any compact smooth manifold M there exists a Morse function, for which each M^a is compact.

Theorem (Milnor)

Let M be a smooth manifold, $K \subset M$ compact, and $k \geq 0$ an integer. Any bounded smooth function $f : M \to \mathbb{R}$ can be uniformly approximated by a Morse function g. Furthermore, for $1 \leq i \leq k$ it is possible to choose g such that the i-th derivatives of g on K uniformly approximate the corresponding derivatives of f.

Applications

There are a number of important applications, including

- Classification of compact 2-manifolds
- h-cobordism Theorem
- Lefschetz Hyperplane Theorem
- Yang-Mills Theory

Openings

There is much active research deriving from Morse Theory:

- Index Theory
- Witten Helffer-Sjöstrand Theory

Further Reading

- John Milnor, Morse Theory
- Raoul Bott, Morse Theory Indomitable
- Edward Witten, Supersymmetry and Morse Theory
- Augustin Banyaga, Lectures on Morse Homology