Witten's Laplacian and the Morse Inequalities*

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Abstract

The Betti numbers $\beta_p = \dim H^p_{dR}(M)$ of a manifold are of great interest, giving strong topological information. The classical Morse Inequalities provide estimates on the Betti numbers in relation to functions $f\colon M \to \mathbb{R}$ with isolated, non-degenerate critical points. In 1982, Witten proved the Morse Inequalities by deforming the Laplacian so as to simplify the computation of its kernel, which leads to the result by utilizing the Hodge Theorem.

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1 Introduction

1.1 Background and Motivation

Given a smooth n-dimensional manifold M, the Betti numbers β_p are defined in terms of the Hodge-deRham cohomology groups $H^p_{dR}(M)$,

$$\beta_p = \dim H^p_{dR}(M) = \dim \frac{\{\alpha \in \Omega^p \colon d\alpha = 0\}}{\{d\beta \colon \beta \in \Omega^{p-1}\}} \tag{1}$$

which give information on the topology of the manifold. The Euler characteristic $\chi(M)$ is defined on manifolds by

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \beta_i. \tag{2}$$

This can be shown to be equivalent to the classical Euler characteristic. A good reference for this, and other algebraic results, is [4].

In the 1920s, Marston Morse initiated the study of functions $f \in C^\infty(M, \mathbb{R})$ with isolated, non-degenerate critical points, which are known as Morse functions. The Morse index m_p of a nondegenerate critical point p is the dimension of the negative eigenspace of the Hessian at p, and the i-th Morse number M_i is the number of critical points with Morse index i. Morse's work lead to the celebrated Morse Inequalities, which provide estimates on the Betti numbers in terms of the Morse numbers.

One approach to determining the Betti numbers is given by Hodge Theory. The Hodge Laplacian $\Delta^p \colon \Omega^p \to \Omega^p$ acting on the space of differential p-forms is defined as $\Delta = d\delta + \delta d$, where $d^p \colon \Omega^p \to \Omega^{p+1}$ denotes the exterior derivative and $\delta^{p+1} \colon \Omega^p \to \Omega^{p-1}$ denotes its formal adjoint. There is the exciting result [2]

Proposition 1.1.

$$\beta_p = \dim H^p_{dR}(M) = \dim \ker \Delta^p \tag{3}$$

which leads to many approaches to determining the Betti numbers. Edward Witten published a proof [8] of the Morse Inequalities following an insight into the kernel of the Laplacian. The main line of thought is as follows:

Defining the twisted exterior derivative $d^p_t \colon \Omega^p \to \Omega^{p+1}$ by

$$d^p_t = e^{-tf} d^p e^{tf} \tag{4}$$

for some Morse function f and denoting its adjoint by $\delta^{p+1}_t \colon \Omega^{p+1} \to \Omega^p$ the Witten Laplacian $\Delta^p_t \colon \Omega^p \to \Omega^p$ is then defined as

$$\Delta_t = d_t \delta_t + \delta_t d_t. \tag{5}$$

It will be shown that (1.1) will still hold for Δ_t, that is

$$\beta_p = \dim \ker \Delta^p_t. \tag{6}$$

Moreover, it will be shown that as $t \to \infty, \omega \in \ker \Delta_t^p$ if and only if $\|df\| \approx 0$; in other words, Witten's Laplacian concentrates the elements of the kernel near the critical points of f. The calculation of the kernel then simplifies dramatically, and the Morse Inequalities can be recovered.

1.2 Outline

The paper will proceed as follows. In the remainder of Section 1, the necessary background in Morse Theory will be presented; in particular the Weak, Strong, and Polynomial Morse Inequalities.

In Section 2, the motivation for Witten's approach to the proof will be considered, recalling the theory of supersymmetric spaces. The Hodge Theorem will be proved, giving the analytic basis for the determination of the Betti numbers by consideration of the kernel of the Hodge Laplacian. Witten's twisted Laplacian will be introduced, and its relation to the Hodge Theorem. Finally, the Witten Laplacian will be expanded and the connection to Morse Theory will become evident.

Section 3 is dedicated to the local coordinate approximation of the Witten Laplacian. First, the Morse Lemma will be proved by applying Hadamard's Lemma, and then an application of of the Morse Lemma will reduce the Witten Laplacian in local coordinates (around critical points of a Morse function) to the Quantum Harmonic Oscillator. The spectrum of the Quantum Harmonic Oscillator will then be computed using the Dirac 'Ladder Operator' method.

The Weak Morse Inequalities will be proved in Section 4, making precise the way in which the local coordinate expansion of the Witten Laplacian provides an upper bound on the Betti numbers. In particular, a significant portion of Section 4 will be spent proving a series of estimates on the Sobolev norms of relevant operators.

The paper will conclude by proving the Strong and Polynomial Morse Inequalities in Section 5. The Polynomial Inequalities are proved directly by applying a twisted exterior derivative (from the definition of the Witten Laplacian) to the elements of the kernel of the Witten Laplacian. The Strong Morse Inequalities are then shown to be equivalent to the Polynomial Inequalities.

1.3 Some Morse Theory

Let $f\colon M \to \mathbb{R}$ be smooth (infinitely differentiable), i.e. $f \in C^\infty(M, \mathbb{R})$.

Definition 1.2.

- *It is said that $p \in M$ is a <u>critical point</u> of f if for some coordinate system $\{x_1, \ldots, x_n\}$ on a neighborhood of p all of the partial derivatives of f vanish. That is, for all $i \in \{1, \ldots, n\}$ it holds that $\frac{\partial f}{\partial x_i}(p) = 0$. Equivalently, the differential of f vanishes at p, i.e. $df(p) = 0$.*

- *The set of critical points of f is denoted $Cr(f) = \{p \in M : df(p) = 0\}$.*

- *A critical point p of f is said to be* <u>nondegenerate</u> *if its Hessian* $H_f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$ *is nonsingular at p.*

- *A function $f \in C^\infty(M, \mathbb{R})$ is called a* <u>Morse function</u> *if all of its critical points are isolated and nondegenerate.*

- *The* <u>Morse index</u> *m_p of a critical point p is defined as the dimension of the negative eigenspace of $H_f(p)$. Equivalently, m_p is the number of linearly independent eigenvectors of $H_f(p)$ with negative eigenvalues.*

- *The i-th* <u>Morse number</u> *$M_i = \#\{p \in M : df(p) = 0, m_p = i\}$ is the number of critical points $p \in M$ with Morse index $m_p = i$. The Morse numbers are invariant under diffeomorphism.*

Morse Theory is the study of Morse functions, and leads to the celebrated Morse Inequalities:

Proposition 1.3 (Morse Inequalities)**.** *Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function. Denoting by β_i the i-th Betti number,*

- *(Weak Morse Inequalities) For all $i \in \{0, \dots, n\}$,*
$$M_i \geq \beta_i. \tag{7}$$

- *(Strong Morse Inequalities) For any $k \in \{0, \dots, n-1\}$,*
$$\sum_{i=0}^{k} (-1)^{i+k} M_i \geq \sum_{i=0}^{k} (-1)^{i+k} \beta_i \tag{8}$$
and also
$$\sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i \beta_i. \tag{9}$$

- *(Polynomial Morse Inequalities) Define for convenience the Morse Polynomial $\mathcal{M}_t = \sum_{i=0}^n M_i t^i$ and the Poincaré Polynomial $\mathcal{P}_t = \sum_{i=0}^n \beta_i t^i$. Then*
$$\mathcal{M}_t - \mathcal{P}_t = (1+t) \sum_{i=0}^{n-1} Q_i t^i \tag{10}$$
for some sequence of non-negative integers Q_i, and any $t \in \mathbb{R}$.

Note that (9) immediately implies
$$\chi(M) = \sum_{i=0}^{n} (-1)^i M_i. \tag{11}$$

These inequalities provide a strong estimate on the Betti numbers, and have several classical proofs; see [5] for the standard approach and more about Morse Theory.

2 Witten's Idea

2.1 Supersymmetry and the Witten Laplacian

Witten begins by recalling the idea of supersymmetric spaces, namely that a Hilbert space $\mathcal{H}$ is called supersymmetric if it has a decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into "bosonic" and "fermionic" spaces, and maps

$$\begin{aligned} &Q_1 \colon \mathcal{H}^+ \to \mathcal{H}^- \\ &Q_2 \colon \mathcal{H}^- \to \mathcal{H}^+ \\ &H, (-1)^F \colon \mathcal{H} \to \mathcal{H} \end{aligned} \tag{12}$$

such that for $i \neq j \in \{1, 2\}, \psi \in \mathcal{H}, \psi^+ \in \mathcal{H}^+, \psi^- \in \mathcal{H}^-$,

$$\begin{aligned} (-1)^F \psi^+ &= \psi^+ \\ (-1)^F \psi^- &= -\psi^- \\ (-1)^F Q_i + Q_i (-1)^F &= 0 \\ Q_i H - H Q_i &= 0 \\ Q_i^2 &= H \\ Q_i Q_j + Q_j Q_i &= 0. \end{aligned} \tag{13}$$

Observe that for the space of mixed differential forms over a Riemannian manifold M there is a decomposition $\Omega = \Omega^+ \oplus \Omega^- = \bigoplus_{p=0}^{n/2} \Omega^{2p} \oplus \bigoplus_{p=0}^{n/2-1} \Omega^{2p+1}$ into even and odd forms. Define

$$\begin{aligned} (-1)^F|_{\Omega^+} &= Id \\ (-1)^F|_{\Omega^-} &= -Id \\ Q_1 &= d + \delta \\ Q_2 &= i(d - \delta) \\ H = \Delta &= d\delta + \delta d \end{aligned} \tag{14}$$

the supersymmetry conditions are met.

He then suggests a generalization of the above; let $f \in C^\infty(M, \mathbb{R})$, and define for $t \in \mathbb{R}$

$$\begin{aligned} d_t &= e^{-tf} d e^{tf} \\ \delta_t &= e^{tf} \delta e^{-tf}. \end{aligned} \tag{15}$$

Immediately these are formally adjoint, since

$$\begin{aligned} \langle d_t \alpha, \beta \rangle &= \langle e^{-tf} d e^{tf} \alpha, \beta \rangle \\ &= \langle d e^{tf} \alpha, e^{-tf} \beta \rangle \\ &= \langle e^{tf} \alpha, \delta e^{-tf} \beta \rangle \\ &= \langle \alpha, e^{tf} \delta e^{-tf} \beta \rangle \\ &= \langle \alpha, \delta_t \beta \rangle \end{aligned} \tag{16}$$

and moreover $d_t^2 = \delta_t^2 = 0$, so analogously defining

$$\begin{aligned} Q_{1t} &= d_t + \delta_t \\ Q_{2t} &= i(d_t - \delta_t) \\ \Delta_t &= d_t\delta_t + \delta_t d_t \end{aligned} \tag{17}$$

the supersymmetry relations are still satisfied over Ω for any t. The operator Δ_t is called the twisted "Witten" Laplacian.

2.2 Hodge Theory

The motivation for the definition of the Witten Laplacian comes from the following.

Let $\Omega = \Omega M$ be the space of differential forms over M, that is the space of smooth sections of the exterior bundle of M (which is written $\Omega M = \Gamma(\Lambda M)$.) Denote by $\Omega^p := \Gamma(\Lambda^p M)$ the space of differential p-forms.

Let $d^p \colon \Omega^p \to \Omega^{p+1}$ be the exterior derivative on p-forms. It has formal adjoint $\delta^{p+1} \colon \Omega^{p+1} \to \Omega^p$, and it can be shown that $\delta^{p+1} = (-1)^{pn+1} * d^{n-p-1} *$ where $* \colon \Omega^p \to \Omega^{n-p}$ is the Hodge Star operator.

Definition 2.1. *The Hodge Laplacian on p-forms $\Delta^p \colon \Omega^p \to \Omega^p$ is given by $\Delta^p = d^{p-1}\delta^p + \delta^{p+1}d^p$. This is often expressed more simply as $\Delta = d\delta + \delta d$.*

For an exposition on differential forms, the exterior derivative, the Hodge Laplacian, and general differential geometry, see [2].

It is said that $\omega \in \Omega$ is harmonic if $\Delta\omega = 0$. There is a decomposition of Ω in terms of harmonic forms and the images of d and δ, which is a deep result in differential geometry:

Proposition 2.2 (Hodge Decomposition)**.** *For a compact Riemannian manifold (M, g) there are the following orthogonal decompositions*

$$\Omega^p = \ker \Delta^p \oplus \operatorname{im} d^{p-1} \oplus \operatorname{im} \delta^{p+1}. \tag{18}$$

For the standard analytic proof of Proposition 2.2 see [7], and for an approach using index theory see [6]. Using this decomposition the Hodge Theorem can be proved.

Theorem 2.3 (Hodge Theorem)**.** *Let (M, g) be a compact, oriented manifold, and denote by $H^p_{dR}(M)$ the p-th deRham cohomology group $\frac{\{\omega\in\Omega^p \colon d\omega=0\}}{\{d\omega \colon \omega\in\Omega^{p-1}\}}$. Then the maps*

$$\begin{aligned} h_p \colon \ker \Delta^p &\to H^p_{dR}(M) \\ \omega &\mapsto [\omega] \end{aligned} \tag{19}$$

are isomorphisms.

Proof. (Hodge Theorem, using Hodge Decomposition)

It is clear that h_p is injective since if $[\omega] = [0]$ then $\omega = d\alpha$ for some $\alpha \in \Omega^{p-1}$. Then

$$0 = \langle \delta\omega, \alpha \rangle = \langle \delta d\alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle \tag{20}$$

which implies $\omega = d\alpha = 0$.

To see that h_p is surjective, let $[\omega] \in H^p_{dR}(M)$ (that is, $d\omega = 0$). It must be shown that there exists a harmonic form γ such that $\omega = \gamma + d\alpha$ for some α. By the Hodge Decomposition (2.2), for some α, β, γ with $\Delta\gamma = 0$,

$$\begin{aligned} \omega &= \gamma + d\alpha + \delta\beta \\ 0 = d\omega &= d\gamma + d^2\alpha + d\delta\beta = d\delta\beta \end{aligned} \tag{21}$$

since harmonic forms are closed. Then

$$\begin{aligned} 0 &= \langle d\delta\beta, \beta \rangle \\ &= \langle \delta\beta, \delta\beta \rangle \\ &= \|\delta\beta\|^2 \end{aligned} \tag{22}$$

so $\delta\beta = 0$ and then in conclusion $\omega = \gamma + d\alpha$, as desired.

□

There is also a heat flow proof of the Hodge Theorem (2.3) that does not require the Hodge Decomposition (2.2). First, recall the heat operator $e^{-t\Delta}$, which uniquely solves the heat equation on a compact manifold M with initial temperature distribution $\omega \in L^2\Omega^k(M)$, i.e.

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_x\right) e^{-t\Delta}\omega = 0 \\ \lim_{t\to 0} e^{-t\Delta}\omega = \omega \end{aligned} \tag{23}$$

is given by

$$(e^{-t\Delta}\omega)(x) = \int_M e(t,x,y)\omega(y)dy \tag{24}$$

where $e(t,x,y) \in C^\infty(\mathbb{R}^+ \times \Lambda^k T^*_x M \times \Lambda^k T^*_y M, \mathbb{R})$ is called the heat kernel. Recall that the heat kernel is symmetric, that is $e(t,x,y) = e(t,y,x)$. For a proof of the existence and properties of the heat kernel on manifolds as well as a broader exposition of the heat operator see for example [6]. The following lemma will be necessary.

Lemma 2.4. $\Delta e^{-t\Delta} = e^{-t\Delta}\Delta$ *and* $de^{-t\Delta} = e^{-t\Delta}d$ *for all smooth forms.*

Proof.

$$\begin{aligned}
\Delta_x(e^{-t\Delta}\omega)(x) &= \Delta_x\left(\int_M e(t,x,y)\omega(y)dy\right) \\
&= \int_M \Delta_x e(t,x,y)\omega(y)dy \\
&= -\int_M \partial_t e(t,x,y)\omega(y)dy \\
&= -\int_M \partial_t e(t,y,x)\omega(y)dy \\
&= \int_M \Delta_y e(t,y,x)\omega(y)dy \\
&= \int_M e(t,y,x)\Delta_y\omega(y)dy \\
&= e^{-t\Delta}(\Delta\omega)(x)
\end{aligned} \tag{25}$$

making use of the fact that Δ is self-adjoint.

To see that $de^{-t\Delta} = e^{-t\Delta}d$, observe that

$$d\Delta = d(d\delta + \delta d) = d^2\delta + d\delta d = d\delta d + d\delta^2 = (d\delta + \delta d)d = \Delta d. \tag{26}$$

Now let $\omega \in \Omega^k$, and observe that $e^{-t\Delta}d\omega$ is a solution to the heat equation with initial condition $d\omega$. Also,

$$\begin{aligned}
\frac{\partial}{\partial t}(de^{-t\Delta}\omega) &= -d\Delta e^{-t\Delta}\omega \\
&= -\Delta(de^{-t\Delta}\omega)
\end{aligned} \tag{27}$$

using (26). This implies $de^{-t\Delta}\omega$ is a solution to the heat equation, also with initial condition $\lim_{t\to 0} de^{-t\Delta}\omega = d\omega$. Notice,

$$\begin{aligned}
\|e^{-t\Delta}d\omega\|_{L^2}^2 &= \int_M \left\|e^{-t\Delta}d\omega\right\|^2 < +\infty \\
\|de^{-t\Delta}\omega\|_{L^2}^2 &= \int_M \left\|de^{-t\Delta}\omega\right\|^2 < +\infty
\end{aligned} \tag{28}$$

because M is compact. Then by the uniqueness of solutions to the heat equation in L^2,

$$de^{-t\Delta} = e^{-t\Delta}d. \tag{29}$$

□

These results allow for a heat flow approach to the proof of the Hodge Theorem.

Proof. (Hodge Theorem, using heat flow.)

Let $\{\omega_i\}$ be an orthonormal basis of Ω^p with $\Delta\omega_i = \lambda_i\omega_i$ where the sequence λ_i is monotone increasing and accumulates only at $+\infty$. This exists since $e^{-t\Delta}\colon H^0 \to H^0$ is a compact, self-adjoint operator, and thus there exists an orthonormal basis $\{\omega_i\}$ of Ω^p of eigenforms of $e^{-t\Delta}$ with eigenvalues $\{e^{-\lambda_i t}\}$ which are monotone decreasing and accumulate only at 0. It can be shown that if $e^{-\lambda_i t}$ is an eigenvalue of $e^{-t\Delta}$, then λ_i is an eigenvalue of Δ, proving the claim.

For any $\omega \in \Omega^p$ it follows that

$$\begin{aligned}
\lim_{t\to\infty} e^{-t\Delta}\omega &= \lim_{t\to\infty} e^{-t\Delta}\sum_i a_i\omega_i \\
&= \lim_{t\to\infty}\sum_i a_i e^{-t\Delta}\omega_i \\
&= \lim_{t\to\infty}\sum_i a_i e^{-\lambda_i t}\omega_i \\
&= \sum_{i=0}^{N} a_i\omega_i
\end{aligned} \tag{30}$$

where $\{\omega_1, \ldots, \omega_N\}$ is an orthonormal basis for $\ker\Delta^p$. Thus a form ω flows to its harmonic component. Now let ω be a closed p-form and consider the class $[\omega] \in H^p_{dR}(M)$. Then

$$\begin{aligned}
e^{-t\Delta}\omega - \omega &= \int_0^t \partial_t(e^{-t\Delta}\omega)dt \\
&= -\int_0^t \Delta e^{-t\Delta}\omega dt \\
&= -\int_0^t e^{-t\Delta}\Delta\omega dt \\
&= -\int_0^t e^{-t\Delta}d\delta\omega dt \\
&= -\int_0^t de^{-t\Delta}\delta\omega dt \\
&= d\left(-\int_0^t e^{-t\Delta}\delta\omega dt\right)
\end{aligned} \tag{31}$$

so $e^{-t\Delta}\omega = \omega + d\left(-\int_0^t e^{-t\Delta}\delta\omega dt\right) \in [\omega]$ for every $t \in \mathbb{R}$.

It must still be shown that $\lim_{t\to\infty} d\int_0^t e^{-t\Delta}\delta\omega = d\Delta^{-1}\delta\omega$ is well defined. Since

$$\Delta^{-1} = \sum_i \frac{1}{\lambda_i}\phi_i \tag{32}$$

it is sufficient to show that $\delta\omega$ is orthogonal to $\ker\Delta$ for ω a closed form. By the Spectral Theorem,

$$\begin{aligned} \omega &= \sum_i \langle \omega, \phi_i \rangle \phi_i \\ \delta\omega &= \sum_i \langle \omega, \phi_i \rangle \delta\phi_i \end{aligned} \tag{33}$$

and $\delta\phi_i = 0$ for any $\phi_i \in \ker\Delta$, so $\delta\omega$ is orthogonal to the kernel of Δ, and $\Delta^{-1}\delta\omega$ is well-defined.

Thus the heat flow takes closed forms to closed forms, the cohomology class of a form is unchanged by the heat flow, and each cohomology class contains at least one harmonic form representative, $\lim_{t\to\infty} e^{-t\Delta}\omega$. Finally, there can be at most one harmonic form representative in each cohomology class since if $\eta_1, \eta_2 \in \ker\Delta$ with $[\eta_1] = [\eta_2]$ then

$$\begin{aligned} \eta_1 &= \eta_2 + d\theta \\ 0 = \delta\eta_1 &= \delta\eta_2 + \delta d\theta \\ &= \delta d\theta \end{aligned} \tag{34}$$

thus

$$\begin{aligned} 0 &= \langle \theta, \delta d\theta \rangle \\ &= \langle d\theta, d\theta \rangle \\ &= \|d\theta\|^2 \end{aligned} \tag{35}$$

so $d\theta = 0$ forcing $\eta_1 = \eta_2$. From this conclude $\ker\Delta^p \cong H^p_{dR}$.

□

From the Hodge Theorem (2.3) there is the immediate corollary (Proposition 1.1)

Corollary 2.5.

$$\beta_p = \dim H^p_{dR}(M) = \dim\ker\Delta^p, \tag{36}$$

which is key to analytic approaches to determining the Betti numbers. However, in general the computation of the kernel of the Hodge Laplacian is highly nontrivial. The central insight in Witten's proof of the Morse Inequalities is to instead consider the kernel of the Witten Laplacian, Δ_t, which is significantly easier to compute as $t \to \infty$.

Lemma 2.6. *For any $t \geq 0$,*

$$\beta_p = \dim\ker\Delta^p_t \tag{37}$$

Proof. Observe that $d_t e^{-tf} = (e^{-tf} d e^{tf}) e^{-tf} = e^{-tf} d$, which implies that $e^{-tf} : \Omega^p \to \Omega^p$ makes the following diagram commute,

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & \Omega^p & \xrightarrow{d} & \Omega^{p+1} & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle e^{-tf}} & & \downarrow{\scriptstyle e^{-tf}} & & \\ \cdots & \xrightarrow{d_t} & \Omega^p & \xrightarrow{d_t} & \Omega^{p+1} & \xrightarrow{d_t} & \cdots \end{array} \tag{38}$$

and thus e^{-tf} is an isomorphism of the co-chain complexes (Ω^p, d^p) and (Ω^p, d_t^p). It follows that $\beta_p = \dim\ker\Delta^p = \dim\ker\Delta_t^p$ for any $t \in \mathbb{R}$, using Corollary 2.5. $\square$

See [4] for a general discussion of cohomology, and [1] for a detailed exposition of Morse cohomology.

2.3 Expansion of the Witten Laplacian

To continue, it will be necessary to compute Δ_t explicitly. For $\omega \in \Omega$,

$$\begin{aligned} d_t\omega &= e^{-tf} d e^{tf}\omega \\ &= e^{-tf}(e^{tf} d\omega + t e^{tf} df \wedge \omega) \\ &= (d + t df\wedge)\omega \end{aligned} \tag{39}$$

and it follows that for $\omega_1, \omega_2 \in \Omega$

$$\begin{aligned} \langle \delta_t\omega_1, \omega_2\rangle &= \langle \omega_1, d_t\omega_2\rangle \\ &= \langle \omega_1, (d + t df\wedge)\omega_2\rangle \\ &= \langle \omega_1, d\omega_2\rangle + \langle \omega_1, t df \wedge \omega_2\rangle \\ &= \langle \delta\omega_1, \omega_2\rangle + \langle t\omega_1, df \wedge \omega_2\rangle \\ &= \langle \delta\omega_1, \omega_2\rangle + \langle t\iota_{\nabla f}\omega_1, \omega_2\rangle \\ &= \langle (\delta + t\iota_{\nabla f})\omega_1, \omega_2\rangle \end{aligned} \tag{40}$$

using the fact that ι_X is adjoint to $X\wedge$ for any p-form X. This implies

$\delta_t = \delta + t\iota_{\nabla f}$ since ω_1, ω_2 were arbitrary. Then

$$\begin{aligned}
\Delta_t &= d_t\delta_t + \delta_t d_t \\
&= (d + tdf\wedge)(\delta + t\iota_{\nabla f}) + (\delta + t\iota_{\nabla f})(d + tdf\wedge) \\
&= d\delta + tdf \wedge \delta + td\iota_{\nabla f} + t^2 df \wedge \iota_{\nabla f} \\
&\qquad + \delta d + t\iota_{\nabla f} d + t\delta df \wedge + t^2 \iota_{\nabla f} df \wedge \\
&= (d\delta + \delta d) + t^2(df \wedge \iota_{\nabla f} + \iota_{\nabla f} df\wedge) + t((\iota_{\nabla f} d + d\iota_{\nabla f}) + (df \wedge \delta + \delta df\wedge)) \\
&= \Delta + t^2(df \wedge (\iota_{\nabla f}) + \iota_{\nabla f}(df\wedge)) + t(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}) \\
&= \Delta\omega + t^2((df \wedge (\iota_{\nabla f})) + ((\iota_{\nabla f} df)\wedge) + (-df \wedge (\iota_{\nabla f}))) + th \\
&= \Delta + t^2\langle df, df\rangle + th \\
&= \Delta + t^2\|df\|^2 + th
\end{aligned} \tag{41}$$

setting $h = \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$. Notice, $\mathcal{L}_{\nabla f}$ is a first order differential operator, and that the leading term of the adjoint of such an operator is the opposite of the leading term of the operator, so h must be a zeroth order differential operator.

Now it follows that for $\omega \in \ker D_t$, as $t \to \infty$ it must be that ω decays rapidly away from the points $\|df\| = 0$; this implies that the zero-eigenforms of Δ_t (as $t \to \infty$) will be concentrated near the critical points of f. Thus it is reasonable to approximate Δ_t by a model operator $\overline{H_t}$ that is the sum of local operators defined on neighborhoods of critical points of f.

Writing $\lambda_p^{(n)}(t)$ for the n-th smallest eigenvalue of Δ_t^p, Witten then observes that on physical grounds there must be an asymptotic expansion as $t \to \infty$

$$\lambda_p^{(n)}(t) \sim t\left(A_p^{(n)} + \frac{B_p^{(n)}}{t} + \frac{C_p^{(n)}}{t^2} + \cdots\right). \tag{42}$$

The equality $\beta_p = \dim\ker\Delta_t^p$ now implies that $\beta_p \le \#\{A_p^{(n)} = 0\}$ since as $t \to \infty$ it is necessary that $A_p^{(n)} = 0$ for for $\lambda_p^{(n)} = 0$.

The rigorous analysis of this asymptotic expansion was first provided by Helffer-Sjöstrand in 1985; here instead the spectrum of Δ_t will be examined directly, following the approach in [9]. First, the kernel of Δ_t restricted to neighborhoods of critical points of f will be computed directly in local coordinates.

3 Local Approximation

3.1 Local Coordinates and the Morse Lemma

Now require $f \in C^\infty(M, \mathbb{R})$ to be Morse. Let $q \in M$ be a critical point of f. Since f is Morse it is possible to find a neighborhood $U \subset M$ of q with coordinates $\{x_1, \ldots, x_n\}$ such that the metric is Euclidean on U, $x_i(q) = 0$, and $f(x) = f(0) - \sum_{i=1}^{m_q} x_i^2 + \sum_{i=m_q+1}^{n} x_i^2$, where m_q is the Morse index of q, that is the dimension of the negative eigenspace of the Hessian of f at q. This can be done by the Morse Lemma.

Theorem 3.1 (Morse Lemma)**.** *Let q be an isolated, nondegenerate critical point for $f \in C^\infty(M, \mathbb{R})$. Then there exists a coordinate system $\{x_1, \ldots, x_n\}$ on a neighborhood U of q such that for $x = (x_1, \ldots, x_n) \in U$,*

$$f(x) = f(q) - \sum_{i=1}^{m_q} x_i^2 + \sum_{i=m_q+1}^{n} x_i^2 \tag{43}$$

where m_q is the Morse index of q.

The proof presented here is adapted from [5], and requires the following lemma.

Lemma 3.2 (Hadamard's Lemma)**.** *Let f in $C^\infty(V, \mathbb{R})$ be a function on a convex neighborhood V of 0 in $\mathbb{R}^n$ with $f(0) = 0$. Then*

$$f(x) = \sum_{i=1}^{n} x_i g_i(x) \tag{44}$$

for some $C^\infty(V, \mathbb{R})$ functions g_i with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof. Using the fundamental theorem of calculus,

$$\begin{aligned} f(x) &= \int_0^1 \frac{d}{dt} f(tx)\ dt \\ &= \int_0^1 \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(tx)\ dt \\ &= \sum_{i=1}^{n} x_i g_i(x) \end{aligned} \tag{45}$$

where $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\ dt$. Then $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0)\ dt$, so the lemma is proved.

□

This is a first order approximation of Taylor's theorem that will hold for any smooth function. It will be used twice in the following proof, taking advantage of the fact that at critical points $\frac{\partial f}{\partial x_i} = 0$.

Proof. (Morse Lemma, Theorem 3.1)

First it will be shown that if there is an expression for f in a neighborhood U of a critical point q of the form

$$f(x) = f(q) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2 \tag{46}$$

that necessarily $\lambda = m_q$. Under the assumption that such an expression exists it follows that for any $1 \leq i, j \leq n$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(q) = \begin{cases} -2 & \text{if } i = j \leq \lambda \\ 2 & \text{if } i = j > \lambda \\ 0 & \text{if } i \neq j \end{cases} \tag{47}$$

and so the matrix for $H_f(q)$ in the basis $\frac{\partial}{\partial x^1}|_q, \ldots \frac{\partial}{\partial x^n}|_q$ is

$$\begin{pmatrix} -2\,\mathrm{Id}_\lambda & 0 \\ 0 & 2\,\mathrm{Id}_{n-\lambda} \end{pmatrix} \tag{48}$$

which shows that there is a subspace of TM_p of dimension λ where $H_f(q)$ is negative definite, and a subspace V of dimension $n - \lambda$ on which it is positive definite. Thus, if $\lambda \neq m_q$ there is a contradiction.

Now it will be shown that there exists such a coordinate system $(x_1, \ldots, x_n)$. Choose the coordinates so that $x_1(q) = \cdots = x_n(q) = 0$. Let $h(x) = f(x) - f(q)$ so that $h(0) = 0$ (abusing notation by identifying points in U with their coordinates). Then by Hadamard's Lemma (3.2) it follows that

$$f(x) - f(q) = h(x) = \sum_{j=1}^{n} x_j g_j(x). \tag{49}$$

Moreover, $g_j(0) = \frac{\partial h}{\partial x_j}(0) = \frac{\partial f}{\partial x_j}(q) = 0$ since f is critical at q. Applying Hadamard's Lemma again,

$$\begin{aligned} g_j(x) &= \sum_{i=1}^{n} x_i k_{ij}(x) \\ f(x) - f(q) &= \sum_{i,j=1}^{n} x_i x_j k_{ij}(x). \end{aligned} \tag{50}$$

Set $\bar{k}_{ij} = \frac{1}{2}(k_{ij} + k_{ji})$. Then $\bar{k}_{ij} = \bar{k}_{ji}$ and $f(x) - f(q) = \sum x_i x_j \bar{k}_{ij}(x)$. Moreover, the matrix $\left(\bar{k}_{ij}(0)\right)_{i,j}$ is equal to $\left(\frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)_{i,j}$ and so is nonsingular.

Proceed by induction. Assume for some $1 \leq r \leq n$ that there exists a coordinate system $(u_1, \ldots, u_n)$ on a neighborhood U_1 of q such that

$$f(u) - f(q) = \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r}^{n} u_i u_j H_{ij}(u) \tag{51}$$

in a neighborhood U of q with the matrices $(H_{ij}(u_1, \ldots, u_n))$ symmetric. By a linear transformation, it is always possible to then find a coordinate system $y = (y_1, \ldots, y_n)$ such that $H_{rr}(0) \neq 0$, as will be shown.

If $H_{ij}(0) = 0$ for all $i, j \geq r$ then f would be degenerate at q, so there must be some $H_{ij}(0) \neq 0$. If $H_{ii}(0) \neq 0$ for some $i \geq r$, a rearrangement of the coordinates will give that $H_{rr}(0) \neq 0$. Otherwise, fix $i', j' \geq r$ such that $H_{i'j'}(0) \neq 0$. Write $y_{i'} = u_{i'} + u_{j'}, y_{j'} = u_{i'} - u_{j'}$, and consider the coordinate system $y = (y_1, \ldots, y_n)$ determined by replacing $u_{i'}, u_{j'}$ by $y_{i'}, y_{j'}$ and setting $y_k = u_k$ for $k \neq i', j'$. Then $\frac{1}{4}(y_{i'}^2 - y_{j'}^2) = u_{i'}u_{j'}$, so in this new coordinate system $f(y) - f(q) = \pm y_1^2 \pm \cdots \pm y_{r-1}^2 + \sum_{i,j \geq r}^n y_i y_j \tilde{H}_{ij}(y)$ where $\tilde{H}_{i'i'}(0) = \frac{1}{4} H_{i'j'} \neq 0$ and $\tilde{H}_{j'j'}(0) = -\frac{1}{4} H_{i''j'}(0) \neq 0$. Rearranging coordinates if necessary, $\tilde{H}_{rr}(0) \neq 0$.

Let $g(u) = |H_{rr}(u)|^{1/2}$. Then g will be a smooth, non-zero function on some neighborhood $U_2 \subset U_1$. Set

$$v_i = \begin{cases} u_i & i \neq r \\ g(u)\left(u_r + \sum_{i>r}^n u_i \frac{H_{ir}(u)}{H_{rr}(u)}\right) & i = r. \end{cases} \tag{52}$$

The Inverse Function Theorem guarantees that there exists some neighborhood $U_3 \subset U_2$ of q on which $v = (v_1, \ldots, v_n)$ will be coordinate functions. Finally,

$$\begin{aligned} f(u) - f(q) &= \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r}^n u_i u_j H_{ij}(u) \\ &= \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + u_r^2 H_{rr}(u) + 2\sum_{i>r}^n u_r u_i H_{ir}(u) + \sum_{i,j>r}^n u_i u_j H_{ij}(u) \\ &= \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + H_{rr}(u)\left(u_r^2 + 2\sum_{i>r}^n u_r u_i \frac{H_{ir}(u)}{H_{rr}(u)} + \left(\sum_{i>r}^n u_i \frac{H_{ir}(u)}{H_{rr}(u)}\right)^2\right) \\ &\quad + \sum_{i,j>r}^n u_i u_j H'_{ij}(u) \\ &= \pm v_1^2 \pm \cdots \pm v_r^2 + \sum_{i,j>r}^n v_i v_j H'_{ij}(v) \end{aligned} \tag{53}$$

which completes the induction. □

It is interesting to note that Witten only requires a local coordinate system that has $f(x) = f(0) + \sum_{i=1}^n \lambda_i x_i^2 + O(x_i^3)$ for some constants λ_i, a Taylor expansion of f about q.

Observe that in these coordinates $df = -\sum_{i=1}^{m_p} 2x_i dx^i + \sum_{i=m_p+1}^n 2x_i dx^i$ so it follows that $\|df\|^2 = 4\sum_{i=1}^n x_i^2$. To express $\mathcal{L}_{\nabla f}$ in local coordinates, observe that $\nabla f = -\sum_{i=1}^{m_q} 2x_i \frac{\partial}{\partial x_i} + \sum_{i=m_q+1}^n 2x_i \frac{\partial}{\partial x_i}$ and that for any $\lambda \in \mathbb{R}$

$$\mathcal{L}_{\lambda x_i \frac{\partial}{\partial x_i}} = \lambda(x_i \mathcal{L}_{\frac{\partial}{\partial x_i}} + dx^i \wedge \iota_{\frac{\partial}{\partial x_i}}) \tag{54}$$

and

$$\begin{aligned}\mathcal{L}^*_{\lambda x_i \frac{\partial}{\partial x_i}} &= \lambda((x_i \mathcal{L}_{\frac{\partial}{\partial x_i}})^* + (dx^i \wedge \iota_{\frac{\partial}{\partial x_i}})^*) \\ &= \lambda(-1 - x_i \mathcal{L}_{\frac{\partial}{\partial x_i}} + dx^i \wedge \iota_{\frac{\partial}{\partial x_i}})\end{aligned} \tag{55}$$

so

$$\begin{aligned}\mathcal{L}_{\lambda x_i \frac{\partial}{\partial x_i}} + \mathcal{L}^*_{\lambda x_i \frac{\partial}{\partial x_i}} &= \lambda(x_i \mathcal{L}_{\frac{\partial}{\partial x_i}} + dx^i \wedge \iota_{\frac{\partial}{\partial x_i}} - 1 - x_i \mathcal{L}_{\frac{\partial}{\partial x_i}} + dx^i \wedge \iota_{\frac{\partial}{\partial x_i}}) \\ &= \lambda(-1 + 2dx^i \wedge \iota_{\frac{\partial}{\partial x_i}}) \\ &= \lambda(dx^i \wedge \iota_{\frac{\partial}{\partial x_i}} - \iota_{\frac{\partial}{\partial x_i}} dx^i \wedge) \\ &= \lambda[dx^i \wedge, \iota_{\frac{\partial}{\partial x_i}}].\end{aligned} \tag{56}$$

Then in conclusion

$$\begin{aligned} h &= \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} \\ &= \mathcal{L}_{-\sum_{i=1}^{m_q} 2x_i \frac{\partial}{\partial x_i} + \sum_{i=m_q+1}^{n} 2x_i \frac{\partial}{\partial x_i}} + \mathcal{L}^*_{-\sum_{i=1}^{m_q} 2x_i \frac{\partial}{\partial x_i} + \sum_{i=m_q+1}^{n} 2x_i \frac{\partial}{\partial x_i}} \\ &= \sum_{i=1}^{m_q} \mathcal{L}_{-2x_i \frac{\partial}{\partial x_i}} + \mathcal{L}^*_{-2x_i \frac{\partial}{\partial x_i}} + \sum_{i=m_q+1}^{n} \mathcal{L}_{2x_i \frac{\partial}{\partial x_i}} + \mathcal{L}^*_{2x_i \frac{\partial}{\partial x_i}} \\ &= 2\left(-\sum_{i=1}^{m_q} [dx^i, \iota_{\frac{\partial}{\partial x_i}}] + \sum_{i=m_q+1}^{n} [dx^i, \iota_{\frac{\partial}{\partial x_i}}]\right) \\ &= \sum_{i=1}^{n} 2\eta_i [dx^i, \iota_{\frac{\partial}{\partial x_i}}]\end{aligned} \tag{57}$$

where $\eta_i = \begin{cases} -1 & i \leq m_q, \\ 1, & i > m_q. \end{cases}$

Observe that for a differential form $\omega = f_I dx^I$ with multi-index I,

$$[dx^i, \iota_{\frac{\partial}{\partial x_i}}]\omega = \begin{cases} \omega & i \in I \\ -\omega & i \notin I \end{cases} \tag{58}$$

so

$$K_i \omega := \eta_i [dx^i, \iota_{\frac{\partial}{\partial x_i}}]\omega = \begin{cases} -\omega & (i \leq m_q \text{ and } i \in I) \text{ or } (i > m_q \text{ and } i \notin I) \\ \omega & \text{otherwise.} \end{cases} \tag{59}$$

In general,

$$\Delta = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j} \frac{\partial}{\partial x_j} \left(g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x_i} \right) \tag{60}$$

However, there is no restriction on the metric in the discussion so far, so by choosing $g = \sum_{i=1}^{n}(dx^i)^2$ at q, it is possible to approximate Δ on U_q by

$$-\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{61}$$

With these local expressions, define the local approximation $H_{q,t}$ on the neighborhood U_q of q to be

$$\begin{aligned} H_{q,t} &= \sum_{i=1}^{n} -\frac{\partial^2}{\partial x_i^2} + 4t^2x_i^2 + 2\eta_i t[dx^i \wedge, \iota_{\frac{\partial}{\partial x_i}}] \\ &= \sum_{i=1}^{n} J_i + 2tK_i \end{aligned} \tag{62}$$

where $J_i = -\frac{\partial^2}{\partial x_i^2} + 4t^2x_i^2$. Since $K_i = \pm 1$, it follows that $[J_i, K_i] = 0$, and thus they can be simultaneously diagonalized. Importantly, their zero-eigenforms are straightforward to compute. The following proposition will be proved in the next section.

Proposition 3.3 (Kernel of $H_{q,t}$)**.** *For any $q \in Cr(f)$, the map $H_{q,t} \colon U_q \to \mathbb{R}$ defined in the local coordinates $\{x_i\}$ given by the Morse lemma on the neighborhood U_q of q by*

$$H_{q,t} = \sum_{i=1}^{n} -\frac{\partial^2}{\partial x_i^2} + 4t^2x_i^2 + 2\eta_i t[dx^i \wedge, \iota_{\frac{\partial}{\partial x_i}}] \tag{63}$$

has kernel of dimension one, and is generated by the eigenform

$$e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q}. \tag{64}$$

and moreover all of the nonzero eigenvalues of $H_{q,t}$ are greater than Ct for some fixed $C > 0$.

3.2 Eigenvalues of the Quantum Harmonic Oscillator

The operator J_i is a scaling of the well-known simple quantum harmonic oscillator from physics; here its eigenvalues will be calculated using the Dirac "ladder operator" method.

Proposition 3.4. *The eigenvalues of J_i are precisely $2t(1+2\lambda)$ for non-negative integers λ.*

Proof. Define the "momentum operator" $p := -i\frac{\partial}{\partial x_i}$ so that $J_i = 4t^2x_i^2 + p^2$. Then define an operator a (and its adjoint $a^\dagger$),

$$\begin{aligned} a &= \sqrt{t}\left(x_i + \frac{i}{2t}p\right) \\ a^\dagger &= \sqrt{t}\left(x_i - \frac{i}{2t}p\right). \end{aligned} \tag{65}$$

Notice the nice relations

$$\begin{aligned} x_i &= \frac{1}{2\sqrt{t}}(a^\dagger + a) \\ p &= i\sqrt{t}(a^\dagger - a). \end{aligned} \tag{66}$$

It will be useful in the following to compute the commutator $[x_i, p]$

$$\begin{aligned} [x_i, p]f = x_i p f - p x_i f &= -i x_i \frac{\partial f}{\partial x_i} + i \frac{\partial}{\partial x_i} x f \\ &= -i x_i \frac{\partial f}{\partial x_i} + i x_i \frac{\partial f}{\partial x_i} + i f \frac{\partial x_i}{\partial x_i} \\ &= i f \end{aligned} \tag{67}$$

so $[x_i, p] = i$.

Define the "number operator" $N = a^\dagger a$. Then

$$\begin{aligned} 2t(1+2N) &= 2t\left(1 + 2a^\dagger a\right) \\ &= 2t\left(1 + 2\sqrt{t}\left(x_i - \frac{i}{2t}p\right)\sqrt{t}\left(x_i + \frac{i}{2t}p\right)\right) \\ &= 2t\left(1 + 2t x_i^2 + i[x_i, p] + \frac{1}{2t}p^2\right) \\ &= 4t^2 x_i^2 + p^2 \\ &= J_i. \end{aligned} \tag{68}$$

Thus an eigenfunction of N is an eigenfunction of J_i. The following commutation relations hold:

$$\begin{aligned} [a, a^\dagger] &= a a^\dagger - a^\dagger a \\ &= \sqrt{t}\left(x_i + \frac{i}{2t}p\right)\sqrt{t}\left(x_i - \frac{i}{2t}p\right) - \sqrt{t}\left(x_i - \frac{i}{2t}p\right)\sqrt{t}\left(x_i + \frac{i}{2t}p\right) \\ &= 2t\frac{i}{2t}[p, x_i] \\ &= 1 \\ [N, a] &= Na - aN \\ &= a^\dagger a a - a a^\dagger a \\ &= -[a, a^\dagger]a \\ &= -a \\ [N, a^\dagger] &= N a^\dagger - a^\dagger N \\ &= a^\dagger a a^\dagger - a^\dagger a^\dagger a \\ &= a^\dagger [a, a^\dagger] \\ &= a^\dagger. \end{aligned} \tag{69}$$

Lemma 3.5. *Let f_λ be an eigenfunction of N with eigenvalue λ. Then*

$$\begin{aligned} Naf_\lambda &= (\lambda - 1)af_\lambda \\ Na^\dagger f_\lambda &= (\lambda + 1)a^\dagger f_\lambda. \end{aligned} \tag{70}$$

These relations motivate referring to $a, a^\dagger$ as the "lowering" and "raising" operators, respectively.

Proof. Directly,

$$\begin{aligned} Naf_\lambda &= (aN + [N, a])f_\lambda \\ &= (aN - a)f_\lambda \\ &= (n - 1)af_\lambda \\ Na^\dagger f_\lambda &= (a^\dagger N + [N, a^\dagger])f_\lambda \\ &= (a^\dagger N + a^\dagger)f_\lambda \\ &= (n + 1)a^\dagger f_\lambda. \end{aligned} \tag{71}$$

□

Lemma 3.6. *The eigenvalues of N are precisely the non-negative integers.*

Proof. First, observe that any eigenvalue of N must be nonnegative. To see this, let f_λ be an eigenfunction with eigenvalue λ. Then

$$\begin{aligned} \lambda\langle f_\lambda, f_\lambda\rangle &= \langle f_\lambda, \lambda f_\lambda\rangle \\ &= \langle f_\lambda, a^\dagger a f_\lambda\rangle \\ &= \langle af_\lambda, af_\lambda\rangle \\ &\geq 0. \end{aligned} \tag{72}$$

Now by induction, for a λ-eigenfunction f_λ, $Na^n f_\lambda = (\lambda - n)a^n f_\lambda$ which implies that $a^n f_\lambda$ is a $\lambda - n$ eigenfunction, unless there is some $0 \leq i \leq n$ such that $\lambda - i = 0$, in which case $a^n f_\lambda = 0$. If λ is not a non-negative integer there is no such i, and then for $n > \lambda$ it would follow that $a^n f_\lambda$ would have negative eigenvalue, which is impossible. □

Finally by equation (68), for a λ-eigenfunction f_λ of N,

$$J_i f_\lambda = 2t(1 + 2N)f_\lambda = 2t(1 + 2\lambda)f_\lambda \tag{73}$$

and thus in conclusion the eigenvalues of J_i are precisely $2t(1 + 2\lambda)$ for non-negative integers λ, concluding the proof of Proposition 3.4.

□

In particular the eigenfunction associated to the smallest eigenvalue of J_i will be of interest.

Proposition 3.7. *The $2t$-eigenfunction of J_i is $e^{-tx_i^2}$.*

Proof. Notice if f_0 is the $2t$-eigenfunction of J_i, then it is the 1-eigenfunction of N. Then $Naf_0 = 0$ so that $af_0 = 0$, an expression which can be solved by separation of variables.

$$\begin{aligned} af_0 &= 0 \\ \sqrt{t}\left(x_i + \frac{i}{2t}p\right) f_0 &= 0 \\ x_i f_0 + \frac{1}{2t}\frac{\partial f_0}{\partial x_i} &= 0 \\ \frac{\partial f_0}{\partial x_i} &= -2tx_i f_0 \\ \ln(f_0) &= -tx_i^2 + C \\ f_0 &= Ce^{-tx_i^2}. \end{aligned} \tag{74}$$

Take $C = 1$ for convenience.

□

This is denoted by $\psi_0 = e^{-tx_i^2}$, which is called the "ground state" (keeping with the physics literature.) Although it will not be needed in the following, here the higher eigenfunctions of J_i will be computed.

Proposition 3.8. *The higher eigenfunctions (or "excited states") ψ_n are given by*

$$\psi_n = \mathcal{H}_n \psi_0 \tag{75}$$

where the Hermite polynomials $\mathcal{H}_n$ are given inductively by

$$\begin{aligned} \mathcal{H}_0(x_i) &= 1 \\ \mathcal{H}_{n+1}(x_i) &= \frac{1}{2\sqrt{t}}\left(4tx_i\mathcal{H}_n(x_i) - \frac{\partial \mathcal{H}_n(x_i)}{\partial x_i}\right). \end{aligned} \tag{76}$$

Proof. The case $n = 0$ is done. Assume for induction that (75) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} \psi_{n+1} &= a^\dagger \psi_n \\ &= \sqrt{t}\left(x_i - \frac{1}{2t}\frac{\partial}{\partial x_i}\right)\left(\mathcal{H}_n(x_i)e^{-tx_i^2}\right) \\ &= \sqrt{t}\left(x_i\mathcal{H}_n(x_i)e^{-tx_i^2} - \frac{1}{2t}e^{-tx_i^2}\frac{\partial \mathcal{H}_n(x_i)}{\partial x_i} - \frac{-2tx_i}{2t}\mathcal{H}_i(x_i)e^{-tx_i^2}\right) \\ &= \sqrt{t}\left(2x_i\mathcal{H}_n(x_i) - \frac{1}{2t}\frac{\partial \mathcal{H}_n(x_i)}{\partial x_i}\right)e^{-tx_i^2} \\ &= \frac{1}{2\sqrt{t}}\left(4tx_i\mathcal{H}_n(x_i) - \frac{\partial \mathcal{H}_n(x_i)}{\partial x_i}\right)\psi_0 \\ &= \mathcal{H}_{n+1}\psi_0 \end{aligned} \tag{77}$$

as desired.

□

This gives a complete accounting of the eigenfunctions of J_i, but omits the question of the domain of the operators $a, a^\dagger$, and N, all of which are unbounded. Thankfully, by (3.8) the eigenfunctions are all in the form of a product of a polynomial with a Gaussian, and so inductively it is the case that ψ_n is in the domain of any finite product of a and $a^\dagger$ for any $n \in \mathbb{N}$. The above proof is adapted from [3].

Now it follows that for an eigenform ω of $H_{q,t}$,

$$H_{q,t}\omega = t\left(2\sum_{i=1}^{n}(1+2j+K_i)\right)\omega \tag{78}$$

(which corresponds to $A_p^{(n)}$ in the asymptotic expansion of $\lambda_p^{(n)}$.) For $H_{q,t}\omega = H_{q,t}g(x)dx^I = 0$, it must be that $j = 0 \implies g(x) = e^{-t|x|^2}$ and that $i \in I$ if and only if $i \leq m_q$. From this it is clear that there is precisely one eigenform generating the kernel of $H_{q,t}$, specifically $e^{-t|x|^2}dx^1 \wedge \cdots \wedge dx^{m_q}$. This proves Proposition 3.3.

4 The Weak Morse Inequalities

There is now a good intuition as to why the Weak Morse Inequalities should hold; by Proposition 3.3, associated to each critical point of f with morse index p there is precisely one element generating the kernel of Δ_t, a p-form. Across the manifold there will be precisely M_p critical points of f with p-forms generating the kernel of Δ_t^p locally. Globally it seems reasonable that an element of the kernel of Δ_t^p must be an element of the local kernels, so M_p is an upper bound on $\dim\ker\Delta_t^p = \beta_p$. Witten argues along these lines. Here is presented a justification by global analysis of the low-lying eigenvalues of Δ_t^p, adapted from [9].

4.1 Proof of the Weak Inequalities

Denote by $E_t^p(c)$ the eigenspace of Δ_t^p with eigenvalues in the interval $[0, c]$. The following key theorem will be proved in the next section.

Theorem 4.1. *For any $c > 0$, there exists a $t_0 > 0$ such that for any $t > t_0$,*

$$\dim E_t^p(c) = M_p \tag{79}$$

where M_p is the p-th Morse number, $0 \leq p \leq n$.

From this, the Weak Morse Inequalities follow directly.

Proof. (Weak Morse Inequalities) Recall that by Lemma 2.6, for any $t \geq 0$

$$\beta_p = \dim\ker\Delta_t^p. \tag{80}$$

Using Theorem 4.1, it is seen that for any $c > 0, 0 \leq p \leq n$ there exists a $t_0 > 0$ such that for $t > t_0$,

$$\dim \ker \Delta_t^p = \dim E_t^p(0) \leq \dim E_t^p(c) = M_p \tag{81}$$

and then since β_p is independent of t,

$$\beta_p \leq M_p \tag{82}$$

$\square$

To prove Theorem 4.1, it will be necessary to utilize the theory of Sobolev spaces of differential forms.

4.2 Sobolev Estimates

For every $q \in Cr(f)$ let U_q be a neighborhood of q with local coordinates $x = \{x_1, \ldots, x_n\}$ as in Proposition 3.3. Without loss of generality assume U_q is an open ball centered at q with radius $4a$, and choose $\gamma\colon \mathbb{R} \to [0,1]$ to be a smooth function such that $\gamma(x) = 1$ when $|x| < a$, and $\gamma(x) = 0$ when $|x| > 2a$. Define

$$\begin{aligned} \alpha_{q,t} &= \left\| \gamma(|x|)e^{-t|x|^2} \right\|^2_{L^2(M)} = \int_{U_q} \gamma(|x|)^2 e^{-2t|x|^2}\ dx^1 \wedge \cdots \wedge dx^n \\ \rho_{q,t} &= \frac{\gamma(|x|)}{\sqrt{\alpha_{q,t}}} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q} \end{aligned} \tag{83}$$

where m_q is the Morse index of q. Then $\rho_{q,t} \in \Omega^{m_q}$ has unit length and is compactly supported in U_q.

Recall the s-th Sobolev space of p-forms $H^s\Lambda^p M$ on a manifold M (see [6] for a discussion.) For a differential p-form u with compact support, $\|u\|_{H^s\Lambda^p(M)} = \|g(u,u)^{1/2}\|_{H^s(M)}$, and in particular $\|u\|_{H^0\Lambda^p(M)} = \|g(u,u)^{1/2}\|_{L^2(M)}$. In the following, denote $\|\cdot\|_i = \|\cdot\|_{H^i(M)}$.

Let E_t be the vector space generated by the set $\{\rho_{q,t}\colon q \in Cr(f)\}$; E_t is a subspace of $H^0(M) = \bigoplus_{i=0}^n H^0\Lambda^i(M)$, since each $\rho_{q,t}$ has finite length and is supported compactly. This implies that there exists an orthogonal splitting

$$H^0(M) = E_t \oplus E_t^\perp \tag{84}$$

where $E_t^\perp$ is the orthogonal complement of E_t in $H^0(M)$. Denote by $p_t, p_t^\perp$ the projection maps from $H^0(M)$ to $E_t, E_t^\perp$, respectively.

In the following, denote by D_t the 'deformed Witten operator'

$$D_t = d_t + \delta_t \tag{85}$$

and observe that

$$D_t^2 = (d_t + \delta_t)(d_t + \delta_t) = d_t^2 + d_t\delta_t + \delta_t d_t + \delta_t^2 = d_t\delta_t + \delta_t d_t = \Delta_t \tag{86}$$

using the fact that $d_t^2 = \delta_t^2 = 0$. This is related to the notion of supersymmetric spaces, as discussed in Section 2.1 (in fact, D_t is just Q_{1t} from equation (17).)

Now, split the deformed Witten operator by the projections, that is define

$$\begin{aligned} D_{t,1} &= p_t D_t p_t \\ D_{t,2} &= p_t D_t p_t^{\perp} \\ D_{t,3} &= p_t^{\perp} D_t p_t \\ D_{t,4} &= p_t^{\perp} D_t p_t^{\perp}. \end{aligned} \tag{87}$$

The following estimates will be critical in the proof of Theorem 4.1.

Proposition 4.2.

1. *For any $t > 0$,*
$$D_{t,1} = 0 \tag{88}$$

2. *There exists a constant $t_1 > 0$ such that for any $s \in E_t^{\perp} \cap H^1(M), s' \in E_t$, and $t > t_1$,*
$$\begin{aligned} \|D_{t,2}s\|_0 &\leq \frac{\|s\|_0}{t} \\ \|D_{t,3}s'\|_0 &\leq \frac{\|s'\|_0}{t} \end{aligned} \tag{89}$$

3. *There exist constants $t_2 > 0$ and $C > 0$ such that for any $s \in E_t^{\perp} \cap H^1(M)$ and $t > t_2$,*
$$\|D_t s\|_0 \leq C\sqrt{t}\|s\|_0 \tag{90}$$

Proof.

1. For any $s \in H^0(M)$ the projection $p_t s$ can be written
$$p_t s = \sum_{q \in Cr(f)} \langle \rho_{q,t}, s \rangle_{H^0(M)} \rho_{q,t}. \tag{91}$$

   Since $D_t = d_t + \delta_t \colon \Omega^i \to \Omega^{i+1} \oplus \Omega^{i-1}$, it holds that for every $q \in Cr(f), \rho_{q,t} \in \Omega^{m_q}$,
$$D_t \left(\langle \rho_{q,t}, s \rangle_{H^0(M)} \rho_{q,t} \right) \in \Omega^{m_q+1} \oplus \Omega^{m_q-1} \tag{92}$$

   will have compact support in U_q. Since $\rho_{q',t} \notin U_q$ for $q' \neq q$ it follows that
$$D_{t,1}s = p_t D_t p_t s = p_t D_t \left(\sum_{q \in Cr(f)} \langle \rho_{q,t}, s \rangle_{H^0(M)} \rho_{q,t} \right) = 0 \tag{93}$$

   proving the claim.

2. Since

$$\begin{aligned}
\langle D_{t,2}s, s'\rangle_{H^0(M)} &= \langle p_t D_t p_t^{\perp} s, s'\rangle_{H^0(M)} \\
&= \langle p_t D_t s, s'\rangle_{H^0(M)} \\
&= \left\langle \sum_{q\in Cr(f)} \langle \rho_{q,t}, D_t s\rangle_{H^0(M)} \rho_{q,t}, s' \right\rangle_{H^0(M)} \\
&= \sum_{q\in Cr(f)} \big\langle \langle \rho_{q,t}, D_t s\rangle_{H^0(M)} \rho_{q,t}, s' \big\rangle_{H^0(M)} \\
&= \sum_{q\in Cr(f)} \langle \rho_{q,t}, D_t s\rangle_{H^0(M)} \langle \rho_{q,t}, s'\rangle_{H^0(M)} \\
&= \sum_{q\in Cr(f)} \langle D_t\rho_{q,t}, s\rangle_{H^0(M)} \langle \rho_{q,t}, s'\rangle_{H^0(M)} \\
&= \sum_{q\in Cr(f)} \langle s, p_t^{\perp} D_t \rho_{q,t}\rangle_{H^0(M)} \langle \rho_{q,t}, s'\rangle_{H^0(M)} \\
&= \sum_{q\in Cr(f)} \langle s, p_t^{\perp} D_t \langle \rho_{q,t}, s'\rangle_{H^0(M)} \rho_{q,t}\rangle_{H^0(M)} \\
&= \left\langle s, p_t^{\perp} D_t \sum_{q\in Cr(f)} \langle \rho_{q,t}, s'\rangle_{H^0(M)} \rho_{q,t} \right\rangle_{H^0(M)} \\
&= \langle s, p_t^{\perp} D_t p_t s'\rangle_{H^0(M)} \\
&= \langle s, D_{t,3} s'\rangle_{H^0(M)}
\end{aligned} \tag{94}$$

which follows because D_t is self-adjoint. This implies that $D_{t,2}$ and $D_{t,3}$ are formally adjoint and thus it is enough to prove the estimate for $D_{t,2}$. Expanding $D_{t,2}$, for any $s \in E_t^{\perp} \cap H^1(M)$,

$$\begin{aligned}
D_{t,2}s &= p_t D_t p_t^{\perp} s \\
&= p_t D_t s \\
&= \sum_{q\in Cr(f)} \langle \rho_{q,t}, D_t s\rangle_{H^0(M)} \rho_{q,t} \\
&= \sum_{q\in Cr(f)} \rho_{q,t} \int_{U_q} \langle \rho_{q,t}, D_t s\rangle \; dv_{U_q} \\
&= \sum_{q\in Cr(f)} \rho_{q,t} \int_{U_q} \langle D_t \rho_{q,t}, s\rangle \; dv_{U_q} \\
&= \sum_{q\in Cr(f)} \rho_{q,t} \int_{U_q} \left\langle D_t \left(\frac{\gamma(|x|)}{\sqrt{\alpha_{q,t}}} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q} \right), s \right\rangle \; dv_{U_q} \\
&= \sum_{q\in Cr(f)} \rho_{q,t} \int_{U_q} \left\langle c \frac{d\gamma(|x|)}{\sqrt{\alpha_{q,t}}} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q}, s \right\rangle \; dv_{U_q}
\end{aligned} \tag{95}$$

using equation (83). Then since $\gamma(|x|)$ is identically one on a neighborhood $V_q \subset\subset U_q$ of q, $d\gamma$ vanishes on this neighborhood and thus,

$$
\begin{aligned}
\|D_{t,2}s\|_0 &\leq \sum_{q\in Cr(f)} \left\| \rho_{q,t} \int_{U_1} \left\langle c\frac{d\gamma(|x|)}{\sqrt{\alpha_{q,t}}} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q}, s \right\rangle \, dv_{U_q} \right\|_0 \\
&\leq \sum_{q\in Cr(f)} \int_{U_q} \left\| c\frac{d\gamma(|x|)}{\sqrt{\alpha_{q,t}}} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q} \right\|_0 \, dv_{U_q} \, \|s\|_0 \\
&\leq \sum_{q\in Cr(f)} \int_{U_q\setminus V_q} Ce^{-2t|x|^2} \left\langle dx^1 \wedge \cdots \wedge dx^{m_q}, dx^1 \wedge \cdots \wedge dx^{m_q} \right\rangle \, dv_{U_q} \, \|s\|_0 \\
&\leq C_1 t^{n/2} e^{-C_2 t} \|s\|_0.
\end{aligned}
\tag{96}
$$

There exists a $t_1 > 0$ such that for $t > t_1$ it will hold that $C_1 t^{n/2} e^{-C_2 t} \leq \frac{1}{t}$, so the claim is proved.

3. For any $0 < b \leq 4a$, denote by $U_q(b), q \in Cr(f)$ the open ball of radius b centered at q, and let $s \in E_t^{\perp} \cap H^1(M)$. The proof will follow in three steps:

   Step 1. Assume $\operatorname{supp}(s) \subset \bigcup_{q\in Cr(f)} U_q(4a)$.

   Step 2. Assume $\operatorname{supp}(s) \subset M \setminus \bigcup_{q\in Cr(f)} U_q(2a)$.

   Step 3. General Case.

   Zhang notes that this approach is due originally to Bismut and Lebeau.

   Step 1. (Assume $\operatorname{supp}(s) \subset \bigcup_{q\in Cr(f)} U_q(4a)$.) Since the Morse Lemma holds on each $U_q(4a)$, it is possible to consider s to be supported on a union of Euclidean spaces $E_q \supset U_q(4a)$, and so the local approximation of D_t in section 3 is exact. For any $t > 0, q \in Cr(f)$, set

$$
\rho'_{q,t} = \left(\frac{t}{\pi}\right)^{n/4} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q} \tag{97}
$$

   and define for any s with $\operatorname{supp}(s) \subset \bigcup_{q\in Cr(f)} U_q(4a)$

$$
p'_t s = \sum_{q\in Cr(f)} \rho'_{q,t} \int_{E_q} \langle \rho'_{q,t}, s\rangle dv_{E_q} \tag{98}
$$

   so the p'_t is the projection of s onto the vector space generated by the $\rho'_{q,t}$. Since $p_t s = 0$ it holds that $\langle \rho_{q,t}, s\rangle = 0$ for all $t > 0, q \in Cr(f)$, and $p'_t s$

can be rewritten as

$$
\begin{aligned}
p_t' s &= \sum_{q \in Cr(f)} \rho_{q,t}' \int_{E_q} \left(\langle \rho_{q,t}', s \rangle - \langle c_t \rho_{q,t}, s \rangle \right) dv_{E_q} \\
&= \sum_{q \in Cr(f)} \rho_{q,t}' \int_{E_q} \langle \rho_{q,t}' - c_t \rho_{q,t}, s \rangle dv_{E_q} \\
&= \sum_{q \in Cr(f)} \rho_{q,t}' \int_{E_q} \langle (1 - \gamma(|x|)) \left(\frac{t}{\pi} \right)^{n/4} e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q}, s \rangle dv_{E_q}
\end{aligned} \tag{99}
$$

where $c_t = \sqrt{\alpha_{q,t}} \left(\frac{t}{\pi} \right)^{n/2}$. Since $\gamma(|x|) = 1$ on a neighborhood $V_q \subset E_q$ of q, there exists a constant $C_3 > 0$ such that for $t \geq 1$,

$$
\begin{aligned}
\|p_t' s\|_0^2 &\leq \sum_{q \in Cr(f)} \|\rho_{q,t}'\|_0^2 \left(\frac{t}{\pi} \right)^{n/2} \|s\|_0^2 \int_{E_q \setminus V_q} \|e^{-t|x|^2} dx^1 \wedge \cdots \wedge dx^{m_q}\|_0^2 \, dv_{E_q} \\
&\leq \frac{C_3}{\sqrt{t}} \|s\|_0^2.
\end{aligned} \tag{100}
$$

Since $D_t p_t' s = 0$, and $s - p_t' s$ is independent of the kernel of D_t it follows that there exist constants $C_4, C_5 > 0$ such that $\|D_t(s - p_t' s)\|_0^2 \geq C_4 t \|s - p_t' s\|_0^2$ (using Proposition 3.3) so for $t \geq 1$

$$
\begin{aligned}
\|D_t s\|_0^2 &= \|D_t(s - p_t' s)\|_0^2 \\
&\geq C_4 t \|s - p_t' s\|_0^2 \\
&\geq \frac{C_4 t}{2} \|s\|_0^2 - \frac{C_5}{\sqrt{t}} \|s\|_0^2
\end{aligned} \tag{101}
$$

from which it directly follows that there exist constants $t_2, C_6 > 0$ such that for all $t > t_2$,

$$
\|D_t s\|_0 \geq C_6 \sqrt{t} \|s\|_0. \tag{102}
$$

Step 2. (Assume $\mathrm{supp}(s) \subset M \setminus \bigcup_{q \in Cr(f)} U_q(2a)$.) Since df is nowhere zero on $V = M \setminus \bigcup_{q \in Cr(f)} U_q(2a)$ and M is compact, there exists a constant $C_7 > 0$ such that $|df|^2 \geq C_6$ on V. Then using equation (41) there exists a $C_8 \geq 0$ such that

$$
\begin{aligned}
\|D_t s\|_0^2 &= \langle D_t s, D_t s \rangle \\
&= \langle \Delta_t s, s \rangle \\
&= \langle (\Delta + t^2 \|df\|^2 + th) s, s \rangle \\
&\geq (C_7 t^2 - C_8 t) \|s\|_0^2
\end{aligned} \tag{103}
$$

so it must be that there exist constants $t_3, C_9 > 0$ such that for $t > t_3$,

$$\|D_t s\|_0 \geq C_9\sqrt{t}\|s\|_0 \tag{104}$$

Step 3. (General Case.) Let $\tilde{\gamma} \in C^\infty(M)$ be such that on each $U_q(4a), \tilde{\gamma}(|x|) = \gamma_q(|x|/2)$ and that $\tilde{\gamma}|_{M\setminus\bigcup_{q\in Cr(f)} U_q(4a)} = 0$. For $s \in E_t^\perp \cap H^1(M)$ it must be that $\tilde{\gamma}s \in E_t^\perp \cap H^1(M)$, so by steps 1 and 2 for any $t \geq t_1 + t_2$, there exists a $C_{10} > 0$ such that

$$\begin{aligned}
\|D_t s\|_0 &\geq \frac{1}{2}\left(\|(1-\tilde{\gamma})D_t s\|_0 + \|\tilde{\gamma}D_t s\|_0\right) \\
&= \frac{1}{2}\left(\|D_t((1-\tilde{\gamma})s) + [D_t,\tilde{\gamma}]s\|_0 + \|D_t(\tilde{\gamma}s) + [\tilde{\gamma}, D_t]s\|_0\right) \\
&\geq \frac{1}{2}\left(\|D_t((1-\tilde{\gamma})s)\|_0 + \|D_t(\tilde{\gamma}s)\|_0\right) + \|[D_t,\tilde{\gamma}]s\|_0 \\
&\geq \frac{\sqrt{t}}{2}\left(C_9\|(1-\tilde{\gamma})s\|_0 + C_6\|\tilde{\gamma}s\|_0\right) - C_{10}\|s\|_0 \\
&\geq C_{11}\sqrt{t}\|s\|_0
\end{aligned} \tag{105}$$

where $C_{11} = \frac{1}{2}\min\{C_6, C_9\}$.

This completes the proof. □

4.3 Proof of Theorem 4.1

For any $c > 0$, denote by $E_t(c)$ the direct sum of the eigenspaces of D_t with eigenvalues lying in $[-c, c]$. $E_t(c)$ is a finite dimensional subspace of $H^0(M)$. Let $P_t(c)$ be the orthogonal projection from $H^0(M)$ to $E_t(c)$. It is necessary to prove a lemma that controls the distance between elements of E_t and their images under $P_t(c)$.

Lemma 4.3. *There exist constants $C, t_3 > 0$ such that for any $t \geq t_3$ and any $\sigma \in E_t$,*

$$\|P_t(c)\sigma - \sigma\|_0 \leq \frac{C}{t}\|\sigma\|_0 \tag{106}$$

Proof. Let $\delta = \{\lambda \in \mathbb{C} \colon |\lambda| = c\}$ be the counterclockwise-oriented circle. Using Proposition 4.2 for any $\lambda \in \delta, t \geq t_1 + t_2$ and $s \in H^1(M)$,

$$\begin{aligned}
2\|(\lambda - D_t)s\|_0 &= \|(\lambda - D_t)s\|_0 + \|(\lambda - D_t)s\|_0 \\
&\geq \|\lambda p_t s - D_{t,2}p_t^\perp s\|_0 + \|\lambda p_t^\perp s - D_{t,3}p_t s - D_{t,4}p_t^\perp s\|_0 \\
&\geq \left|\|\lambda p_t s\|_0 - \|D_{t,2}p_t^\perp\|_0\right| + \left|\|\lambda p_t^\perp s - D_{t,4}p_t^\perp s\|_0 - \|D_{t,3}p_t s\|_0\right| \\
&\geq \left|c\|p_t s\|_0 - \frac{1}{t}\|p_t^\perp s\|_0\right| + \left|\left|\|D_{t,4}p_t^\perp s\|_0 - \|\lambda p_t^\perp s\|_0\right| - \frac{1}{t}\|p_t s\|_0\right| \\
&\geq \left(c - \frac{1}{t}\right)\|p_t s\|_0 + \left(C\sqrt{t} - c - \frac{1}{t}\right)\|p_t^\perp s\|_0
\end{aligned} \tag{107}$$

from which it follows that there exist constants $t_3, C' > 0$ such that for any $t \geq t_3, \lambda \in \delta$, and $s \in H^1(M)$,

$$\|(\lambda - D_t)s\|_0 \geq C'\|s\|_0 \tag{108}$$

thus for any $t \geq t_3, \lambda \in \delta$

$$\lambda - D_t \colon H^1(M) \to H^0(M) \tag{109}$$

is invertible and the resolvent $(\lambda - D_t)^{-1}$ is well-defined. By the spectral theorem for compact operators,

$$P_t(c)\sigma - \sigma = \frac{1}{2\pi i}\int_\delta \left((\lambda - D_t)^{-1} - \lambda^{-1}\right)\sigma \; d\lambda \tag{110}$$

Now it follows directly from Proposition 4.2(1) that

$$\left((\lambda - D_t)^{-1} - \lambda^{-1}\right)\sigma = \lambda^{-1}(\lambda - D_t)^{-1}D_{3,t}\sigma. \tag{111}$$

From (108) and Proposition 4.2(2) it holds that

$$\|(\lambda - D_t)^{-1}D_{t,3}\sigma\|_0 \leq \frac{1}{C'}\|D_{t,3}\sigma\|_0 \leq \frac{1}{C't}\|\sigma\|_0 \tag{112}$$

so finally there exists $C > 0$ such that

$$\|P_t(c)\sigma - \sigma\|_0 \leq \frac{C}{t}\|\sigma\|_0 \tag{113}$$

for all $t \geq t_3$. □

Now, using the estimates in Proposition 4.2 and Lemma 4.3 it is possible to prove Theorem 4.1, completing the proof of the Weak Morse Inequalities.

Proof. (Theorem 4.1) Applying Lemma 4.3 to the $\rho_{q,t}$ for $q \in Cr(f)$, it is clear that there exists a $t_4 > 0$ such that for $t \geq t_4$ the $P_t(c)\rho_{q,t}$ are linearly independent, and thus

$$\dim E_t(c) \geq \dim E_t. \tag{114}$$

It remains to be shown that $\dim E_t(c) = \dim E_t$. Assume for contradiction that $\dim E_t(c) > \dim E_t$. Then there should exist some nonzero $s \in E_t(c)$ orthogonal to $P_t(c)E_t$, that is

$$\langle s, P_t(c)\rho_{q,t}\rangle_0 = 0 \tag{115}$$

for all $q \in Cr(f)$. Then it is possible to write

$$\begin{aligned} p_t s &= \sum_{q\in Cr(f)} \langle s, \rho_{q,t}\rangle_0 \rho_{q,t} \\ &= \sum_{q\in Cr(f)} \langle s, \rho_{q,t}\rangle_0 \rho_{q,t} - \sum_{q\in Cr(f)} \langle s, P_t(c)\rho_{q,t}\rangle_0 P_t(c)\rho_{q,t} \\ &= \sum_{q\in Cr(f)} \langle s, \rho_{q,t}\rangle_0 (\rho_{q,t} - P_t(c)\rho_{q,t}) \\ &\qquad + \sum_{q\in Cr(f)} \langle s, \rho_{q,t} - P_t(c)\rho_{q,t}\rangle_0 P_t(c)\rho_{q,t} \end{aligned} \tag{116}$$

Then, applying Lemma 4.3, there exists a $C > 0$ such that for $t \geq t_3$,

$$\begin{aligned}
\|p_t s\|_0 \leq & \sum_{q \in Cr(f)} \|\langle s, \rho_{q,t}\rangle_0 (\rho_{q,t} - P_t(c)\rho_{q,t})\|_0 \\
& \qquad + \sum_{q \in Cr(f)} \|\langle s, \rho_{q,t} - P_t(c)\rho_{q,t}\rangle_0 P_t(c)\rho_{q,t}\|_0 \\
& \leq \sum_{q \in Cr(f)} (\|s\|_0 \|\rho_{q,t}\|_0 + \|s\|_0 \|P_t(c)\rho_{q,t}\|_0) \|P_t(c)\rho_{q,t} - \rho_{q,t}\|_0 \\
& \leq \frac{C}{t}\|s\|_0
\end{aligned} \tag{117}$$

and moreover there exists a $C' > 0$ such that when t is large enough,

$$\|p_t^\perp s\|_0 \geq \|s\|_0 - \|p_t s\|_0 \geq \left(1 - \frac{C}{t}\right)\|s\|_0 \geq C'\|s\|_0. \tag{118}$$

Using this and Proposition 4.2,

$$\begin{aligned}
CC'\sqrt{t}\|s\|_0 &\leq \|D_t p_t^\perp s\|_0 \\
&= \|D_t s - D_t p_t s\|_0 \\
&\leq \|D_t s\|_0 + \|D_{t,3} s\|_0 \\
&\leq \|D_t s\|_0 + \frac{1}{t}\|s\|_0
\end{aligned} \tag{119}$$

so that

$$\|D_t s\|_0 \geq \left(CC'\sqrt{t} - \frac{1}{t}\right)\|s\|_0 \geq \frac{CC't^{3/2} - 1}{t}\|s\|_0 \tag{120}$$

which contradicts with the assumption that s is a nonzero element of $E_t(c)$ when t is sufficiently large. Thus

$$\dim E_t(c) = \dim E_t = \sum_{i=0}^{n} M_i \tag{121}$$

and moreover $E_t(c)$ is generated by the $P_t(c)\rho_{q,t}$. Now, for any integer i with $0 \leq i \leq n$, denote by Q_i the orthogonal projection from $H^0(M)$ to the L^2 completion of $\Omega^i(M)$. Since Δ_t preserves the $\mathbb{Z}$-grading of $\Omega(M)$ (that is, $\Delta\colon \Omega^i(M) \to \Omega^i(M)$ for any $0 \leq i \leq n$) it holds that for $s \in E_t(c)$ with eigenvalue $\mu \in [-c, c]$,

$$\Delta_t Q_i s = Q_i \Delta_t s = \mu^2 Q_i s \tag{122}$$

That is, $Q_i s$ is an eigenform of Δ_t with eigenvalue μ^2, and so all that remains to be shown is that for t large enough, $\dim Q_i E_t(c) = M_i$. By Lemma 4.3, for $q \in Cr(f)$ there exists a $C > 0$ such that for t large enough,

$$\|Q_{m_q} P_t(c)\rho_{q,t} - \rho_{q,t}\|_0 \leq \frac{C}{t} \tag{123}$$

thus for t large enough, the forms $Q_{m_q} P_t(c)\rho_{q,t}$ are linearly independent and thus $\dim Q_i E_t(c) \geq M_i$ for each $0 \leq i \leq n$. However,

$$\sum_{i=0}^{n} \dim Q_i E_t(c) \leq \dim E_t(c) = \sum_{i=0}^{n} M_i. \tag{124}$$

If for any $0 \leq i \leq n$ it held that $\dim Q_i E_t(c) > M_i$ it would contradict (124), and so

$$\dim Q_i E_t(c) = M_i \tag{125}$$

for sufficiently large t.

□

5 The Strong Morse Inequalities

Recall that it is still necessary to prove the Polynomial and Strong Morse Inequalities. Witten proves directly the Polynomial Morse Inequalities, the Strong will follow by equivalence.

5.1 Polynomial Morse Inequalities

Theorem 5.1 (Polynomial Morse Inequalities)**.**

$$\sum_{p=0}^{n} M_p t^p - \sum_{p=0}^{n} \beta_p t^p = (1+t) \sum_{p=0}^{n} Q_p t^p \tag{126}$$

for some non-negative integers Q_p and $t \in \mathbb{R}$.

The following proof is adapted from [1]; see also [9].

Proof. Let $C_p(f)$ be the the free abelian group generated by the critical points $q \in M$ with Morse index p. Denote by d_t^p the restriction of d_t above to $C_p(f)$ (identifying critical points with the associated element of the kernel of Δ_t). Then $d_t^p \colon C_p(f) \to C_{p+1}(f)$ is a co-boundary operator for the Morse-Smale-Witten co-chain

$$0 \to C_1(f) \to \cdots \to C_n(f) \to 0. \tag{127}$$

There is the exact sequence

$$0 \to \ker d_t^p \to C_p(f) \xrightarrow{d_t^p} \operatorname{im} d_t^p \to 0 \tag{128}$$

which implies that $M_p = \operatorname{rank} C_p(f) = \operatorname{rank} \ker d_t^p + \operatorname{rank} \operatorname{im} d_t^p$. The sequence

$$0 \to \operatorname{im} d_t^{p-1} \to \ker d_t^p \to H_k(C_*(f), d_t^*) \to 0 \tag{129}$$

is also exact $\left(\text{where } H_k(C_*(f), d_t^*) = \frac{\ker d_t^p}{\operatorname{im} d_t^{p-1}}\right)$, which implies

$$\beta_p = \operatorname{rank} H_k(C_*(f), d_t^*) = \operatorname{rank} \ker d_t^p - \operatorname{rank} \operatorname{im} d_t^{p-1}, \tag{130}$$

(recalling the commutative diagram 38.) Then,

$$\begin{aligned}
\sum_{p=0}^{n} M_p t^p - \sum_{p=0}^{n} \beta_p t^p &= \sum_{p=0}^{n} (\operatorname{rank}\ker d_t^p + \operatorname{rank}\operatorname{im} d_t^p) t^p - \sum_{p=0}^{n} (\operatorname{rank}\ker d_t^p - \operatorname{rank}\operatorname{im} d_t^{p-1}) t^p \\
&= \sum_{p=0}^{n} (\operatorname{rank}\operatorname{im} d_t^p + \operatorname{rank}\operatorname{im} d_t^{p-1}) t^p \\
&= \sum_{p=0}^{n} (M_p - \operatorname{rank}\ker d_t^p + M_{p-1} - \operatorname{rank}\ker d_t^{p-1}) t^p \\
&= \sum_{p=0}^{n} (M_p - \operatorname{rank}\ker d_t^p) t^p + \sum_{p=1}^{n} (M_{p-1} - \operatorname{rank}\ker d_t^{p-1}) t^p \\
&= \sum_{p=1}^{n} (M_p - \operatorname{rank}\ker d_t^p) t^p + t \sum_{p=0}^{n-1} (M_p - \operatorname{rank}\ker d_t^p) t^p \\
&= (1+t) \sum_{p=0}^{n-1} (M_p - \operatorname{rank}\ker d_t^p) t^p \\
&= (1+t) \sum_{p=0}^{n-1} Q_p t^p
\end{aligned} \tag{131}$$

where $Q_p = M_p - \operatorname{rank}\ker d_t^p \geq 0$ (and in particular $M_n - \operatorname{rank}\ker d_t^n = 0$.) □

It was known before Witten's proof that the existence of such a boundary operator was equivalent to the Polynomial Morse Inequalities but Witten's approach gives a canonical form d_t^p for the operator (up to the choice of Riemannian metric.)

5.2 Strong Morse Inequalities

Theorem 5.2. *The Strong Morse Inequalities (8) and (9) are equivalent to the Polynomial Morse Inequalities (10).*

Proof. Assuming the Strong Morse Inequalities, it follows that

$$\mathcal{M}_{-1} = \sum_{i=0}^{n} M_i (-1)^i = \sum_{i=0}^{n} \beta_i (-1)^i = \mathcal{P}_{-1} \tag{132}$$

which implies that $\mathcal{M}_t - \mathcal{P}_t$ is divisible by $1+t$. Equivalently,

$$\mathcal{M}_t - \mathcal{P}_t = (1+t) \sum_{i=0}^{n-1} Q_i t^i. \tag{133}$$

It is clear that $Q_i \in \mathbb{Z}$, since $M_i, \beta_i \in \mathbb{Z}$. It remains to be shown that $Q_i \geq 0$. Observe first that by equating the constant terms,

$$M_0 = B_0 + Q_0. \tag{134}$$

Doing the same for the coefficients of t,

$$\begin{aligned} M_1 &= B_1 + Q_1 + Q_0 \\ &= B_1 + Q_1 + (M_0 - B_0) \\ M_1 - M_0 &= B_1 - B_0 + Q_1, \end{aligned} \tag{135}$$

and continuing in this manner it follows that

$$\sum_{i=0}^{k} (-1)^{i+k} M_i = \sum_{i=0}^{k} (-1)^{i+k} \beta_i + Q_k \tag{136}$$

for any $k \in \{0, 1, \ldots, n-1\}$. Then from the (8), $Q_i \geq 0$, so the Strong Morse Inequalities imply the Polynomial Morse Inequalities.

Now assume the Polynomial Morse Inequalities. By the same argument as in (135), equation (136) will hold with $Q_k \geq 0$. Then

$$\sum_{i=0}^{k} (-1)^{i+k} M_i \geq \sum_{i=0}^{k} (-1)^{i+k} \beta_i \tag{137}$$

for every $k \in \{0, 1, \ldots, n-1\}$, which is equation (8). Finally, letting $t = -1$ in (10) it follows that

$$\sum_{i=0}^{n} (-1)^{i} M_i = \sum_{i=0}^{n} (-1)^{i} \beta_i \tag{138}$$

which is equation (9), completing the proof. □

The above proof is adapted from [1], which is an excellent reference on Morse Theory.

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